Cat_∞ and Marked Simplicial Sets

Ramkumar Ramachandra

Université de Paris

16th April 2021

Table of Contents

The definition of Cat_∞

The simplicial category $\mathsf{Cat}^{\Delta}_\infty$ is defined as follows:

- The objects are small ∞-categories.
- Given two ∞-categories $\mathscr{C}$ and $\mathscr{D}$, we define $\mathrm{Map}_{\mathsf{Cat}^{\Delta}_\infty}(\mathscr{C}, \mathscr{D})$ to be the largest Kan complex contained in the ∞-cateogry $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$.

Cat_∞ denotes the simplicial nerve $\mathrm{N}(\mathsf{Cat}^{\Delta}_\infty)$, and is referred to as the ∞-category of small ∞-categories whose mapping spaces are Kan complexes, and composition is strictly associative.

Table of Contents

Motivation

The Joyal model structure on $\mathsf{Set}_{\mathbf{\Delta}}$ is not compatible with the usual simplicial structure. We will introduce marked simplicial sets $\mathsf{Set}^+_{\mathbf{\Delta}}$ as a remedy to the problem, so that we obtain an equivalance of simplicial categories $\mathsf{Cat}^{\mathbf{\Delta}}_\infty \simeq (\mathsf{Set}^+_{\mathbf{\Delta}})^\circ$.

Definition

A marked simplicial set is a pair $(X, \mathcal{E})$, where X is a simplicial set and $\mathcal{E}$ is a set of edges of X, which contains every degenerate edge. We will say that X is marked if it belongs to $\mathcal{E}$. A morphism $(X, \mathcal{E}) \to (X', \mathcal{E}')$ is a map $f : X \to X'$ having the property that $f(\mathcal{E}) \subseteq \mathcal{E}'$.

$S^\sharp$ and $S^\flat$

The two extreme cases of marked simplicial sets are:

- $S^\sharp = (S, S_1)$ denotes the marked simplicial set in which every edge of S is marked.
- $S^\flat = (S, s_0(S_0))$ denotes the marked simplicial set in which only the degerate edges of S are marked

We let $(\mathrm{Set}^+_\Delta)_{/S}$ denote the category which might otherwise be denoted as $(\mathrm{Set}^+_\Delta)_{/S^\sharp}$.

We will soon introduce the cartesian model structure on $(\mathrm{Set}^+_\Delta)_{/S}$, and in particular see that each $(\mathrm{Set}^+_\Delta)_{/S}$ is a simplicial model category whose fibrant objects are the cartesian fibrations $X \to S$.

$S^\sharp$ and $S^\flat$

The two extreme cases of marked simplicial sets are:

- $S^\sharp = (S, S_1)$ denotes the marked simplicial set in which every edge of S is marked.
- $S^\flat = (S, s_0(S_0))$ denotes the marked simplicial set in which only the degerate edges of S are marked

We let $(\text{Set}^+_{\Delta})_{/S}$ denote the category which might otherwise be denoted as $(\text{Set}^+_{\Delta})_{/S^\sharp}$.

We will soon introduce the cartesian model structure on $(\text{Set}^+_{\Delta})_{/S}$, and in particular see that each $(\text{Set}^+_{\Delta})_{/S}$ is a simplicial model category whose fibrant objects are the cartesian fibrations $X \to S$.

$S^\sharp$ and $S^\flat$

The two extreme cases of marked simplicial sets are:

- $S^\sharp = (S, S_1)$ denotes the marked simplicial set in which every edge of S is marked.
- $S^\flat = (S, s_0(S_0))$ denotes the marked simplicial set in which only the degerate edges of S are marked

We let $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ denote the category which might otherwise be denoted as $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S^\sharp}$.

We will soon introduce the cartesian model structure on $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$, and in particular see that each $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ is a simplicial model category whose fibrant objects are the cartesian fibrations $X \to S$.

$S^\sharp$ and $S^\flat$

The two extreme cases of marked simplicial sets are:

- $S^\sharp = (S, S_1)$ denotes the marked simplicial set in which every edge of S is marked.
- $S^\flat = (S, s_0(S_0))$ denotes the marked simplicial set in which only the degerate edges of S are marked

We let $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S}$ denote the category which might otherwise be denoted as $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S^\sharp}$.
We will soon introduce the cartesian model structure on $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S}$, and in particular see that each $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S}$ is a simplicial model category whose fibrant objects are the cartesian fibrations $X \to S$.

Table of Contents

Definition

The class of marked anodyne morphisms in $\mathrm{Set}^+_{\mathbf{\Delta}}$ is the smallest weakly saturated class of morphisms with the following properties:

(1) For each $0 < i < n$, the inclusion $(\Lambda^n_i)^\flat \subseteq (\mathbf{\Delta}^n)^\flat$ is marked anodyne.

(2) For every $n > 0$ the inclusion

$$(\Lambda^n_n, \mathcal{E} \cap (\Lambda^n_n)_1) \subseteq (\mathbf{\Delta}^n, \mathcal{E})$$

is marked anodyne, where $\mathcal{E}$ denotes the set of all degenerate deges of $\mathbf{\Delta}^n$, together with the final edge $\mathbf{\Delta}^{\{n-1,n\}}$.

Definition

The class of marked anodyne morphisms in $\mathrm{Set}^+_{\mathbf{\Delta}}$ is the smallest weakly saturated class of morphisms with the following properties:

(1) For each $0 < i < n$, the inclusion $(\Lambda^n_i)^\flat \subseteq (\mathbf{\Delta}^n)^\flat$ is marked anodyne.

(2) For every $n > 0$ the inclusion

$$(\Lambda^n_n, \mathcal{E} \cap (\Lambda^n_n)_1) \subseteq (\mathbf{\Delta}^n, \mathcal{E})$$

is marked anodyne, where $\mathcal{E}$ denotes the set of all degenerate deges of $\mathbf{\Delta}^n$, together with the final edge $\mathbf{\Delta}^{\{n-1,n\}}$.

(3) The inclusion

$$(\Lambda^2_1)^\sharp \coprod_{(\Lambda^2_1)^\flat} (\mathbf{\Delta}^2)^\flat \to (\mathbf{\Delta}^2)^\sharp$$

is marked anodyne.

(4) For every Kan complex K, the map $K^\flat \to K^\sharp$ is marked anodyne.

(3) The inclusion

$$(\Lambda^2_1)^\sharp \coprod_{(\Lambda^2_1)^\flat} (\boldsymbol{\Delta}^2)^\flat \to (\boldsymbol{\Delta}^2)^\sharp$$

is marked anodyne.

(4) For every Kan complex K, the map $K^\flat \to K^\sharp$ is marked anodyne.

Table of Contents

Groundwork

Let S be a simplicial set. In this section, the goal is to define the cartesian model structure on the category $(\mathsf{Set}_{\mathbf{\Delta}}^+)_{/S}$ of marked simplicial sets over S. The ultimate goal is to prove that the fibrant objects of $(\mathsf{Set}_{\mathbf{\Delta}}^+)_{/S}$ correspond precisely to cartesian fibrations $X \to S$ and that they encode contravariant functors from S into the ∞-category $\mathsf{Cat}_\infty^{\mathbf{\Delta}}$.

First of all, the category $\mathrm{Set}_{\mathbf{\Delta}}^+$ is cartesian-closed. This means that for any two objects $X, Y \in \mathrm{Set}_{\mathbf{\Delta}}^+$, there exists an internal mapping object Y^X equipped with an evaluation map $Y^X \times X \to Y$ which induces bijections

$$\mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}^+}(Z, Y^X) \to \mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}^+}(Z \times X, Y)$$

for every $Z \in \mathrm{Set}_{\mathbf{\Delta}}^+$.

Let $\mathrm{Map}^\flat(X, Y)$ denote the underlying simplicial set of Y^X and $\mathrm{Map}^\sharp(X, Y)$ the simplicial subset of $\mathrm{Map}^\flat(X, Y)$ consisting of all simplices $\sigma \subset \mathrm{Map}^\flat(X, Y)$ such that every edge of σ is a marked edge of Y^X.

First of all, the category $\mathrm{Set}^+_{\mathbf{\Delta}}$ is cartesian-closed. This means that for any two objects $X, Y \in \mathrm{Set}^+_{\mathbf{\Delta}}$, there exists an internal mapping object Y^X equipped with an evaluation map $Y^X \times X \to Y$ which induces bijections

$$\mathrm{Hom}_{\mathrm{Set}^+_{\mathbf{\Delta}}}(Z, Y^X) \to \mathrm{Hom}_{\mathrm{Set}^+_{\mathbf{\Delta}}}(Z \times X, Y)$$

for every $Z \in \mathrm{Set}^+_{\mathbf{\Delta}}$.
Let $\mathrm{Map}^\flat(X, Y)$ denote the underlying simplicial set of Y^X and $\mathrm{Map}^\sharp(X, Y)$ the simplicial subset of $\mathrm{Map}^\flat(X, Y)$ consisting of all simplices $\sigma \subset \mathrm{Map}^\flat(X, Y)$ such that every edge of σ is a marked edge of Y^X.

These simplicial sets can also be described by the properties:

$$\mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}}(K, \mathrm{Map}^\flat(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}^+_{\mathbf{\Delta}}}(K^\flat \times X, Y)$$
$$\mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}}(K, \mathrm{Map}^\sharp(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}^+_{\mathbf{\Delta}}}(K^\sharp \times X, Y)$$

If X and Y are objects of $(\mathrm{Set}_{\mathbf{\Delta}})^+_{/S}$, then we let

$$\mathrm{Map}^\sharp_S(X, Y) \subset \mathrm{Map}^\sharp(X, Y)$$
$$\mathrm{Map}^\flat_S(X, Y) \subset \mathrm{Map}^\flat(X, Y)$$

denote the simplicial subsets classifying those maps compatible with the projections to S. For instance, in the case where $X \in (\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ and $P : Y \to S$ is a cartesian fibration, then $\mathrm{Map}^\flat_S(X, Y^\natural)$ is an ∞-category and $\mathrm{Map}^\sharp_S(X, Y^\natural)$ is the largest Kan complex contained in $\mathrm{Map}^\flat_S(X, Y^\natural)$.

These simplicial sets can also be described by the properties:

$$\mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}}(K, \mathrm{Map}^\flat(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}^+}(K^\flat \times X, Y)$$
$$\mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}}(K, \mathrm{Map}^\sharp(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}_{\mathbf{\Delta}}^+}(K^\sharp \times X, Y)$$

If X and Y are objects of $(\mathrm{Set}_{\mathbf{\Delta}})^+_{/S}$, then we let

$$\mathrm{Map}^\sharp_S(X, Y) \subset \mathrm{Map}^\sharp(X, Y)$$
$$\mathrm{Map}^\flat_S(X, Y) \subset \mathrm{Map}^\flat(X, Y)$$

denote the simplicial subsets classifying those maps compatible with the projections to S. For instance, in the case where $X \in (\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ and $P : Y \to S$ is a cartesian fibration, then $\mathrm{Map}^\flat_S(X, Y^\natural)$ is an ∞-category and $\mathrm{Map}^\sharp_S(X, Y^\natural)$ is the largest Kan complex contained in $\mathrm{Map}^\flat_S(X, Y^\natural)$.

Cartesian equivalences

Let S be a simplicial set. And let $p : X \to Y$ be a morphism in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$. p is a cartesian equivalence if it satisfies either of the following equivalent conditions:

(1) For every cartesian fibration $Z \to S$, the induced map

$$\mathrm{Map}^\flat_S(Y, Z^\natural) \to \mathrm{Map}^\flat_S(X, Z^\natural)$$

is an equivalence of ∞-categories.

(2) For every cartesian fibration $Z \to S$, the induced map

$$\mathrm{Map}_S^\sharp(Y, Z^\natural) \to \mathrm{Map}_S^\sharp(X, Z^\natural)$$

is a homotopy equivalence of Kan complexes.

Let's take $f : X \to Y$ to be a morphism in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ which is marked anodyne when regarded as a map of marked simplicial sets. Since the smash product of f with any inclusion $A^\flat \subset B^\flat$ is also marked anodyne, we deduce that the map

$$\phi : \mathrm{Map}^\flat_S(Y, Z^\natural) \to \mathrm{Map}^\flat_S(X, Z^\natural)$$

is a trivial fibration for every cartesian fibration $Z \to S$, which makes f a cartesian equivalence.

Strong homotopy in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$

$X, Y \in (\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ for S a simplicial set. We say that a pair of morphisms $f, g : X \to Y$ are strongly homotopic if there exists a contractible Kan complex K and a map $K \to \mathrm{Map}^\flat_S(X, Y)$ whose image contains both of the vertices f and g. If $Y = Z^\natural$, where $Z \to S$ is a cartesian fibration, then this just means that f and g are equivalent when viewed as objects of the ∞-category $\mathrm{Map}^\flat_S(X, Y)$.

Let $X \xrightarrow{p} Y \xrightarrow{q} S$ be a diagram of simplicial sets, where both q and $q \circ p$ are cartesian fibrations. The following assertions are equivalent:

(1) The map p induces a cartesian equivalence $X^\natural \to Y^\natural$ in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(2) There exists a map $r : Y \to X$ which is a strong homotopy inverse to p, in the sense that $p \circ r$ and $r \circ p$ are both strongly homotopic to the identity.

(3) The map p induces a categorical quivalence $X_s \to Y_s$ for each vertex s of S.

Let $X \xrightarrow{p} Y \xrightarrow{q} S$ be a diagram of simplicial sets, where both q and $q \circ p$ are cartesian fibrations. The following assertions are equivalent:

(1) The map p induces a cartesian equivalence $X^\natural \to Y^\natural$ in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(2) There exists a map $r : Y \to X$ which is a strong homotopy inverse to p, in the sense that $p \circ r$ and $r \circ p$ are both strongly homotopic to the identity.

(3) The map p induces a categorical quivalence $X_s \to Y_s$ for each vertex s of S.

Let $X \xrightarrow{p} Y \xrightarrow{q} S$ be a diagram of simplicial sets, where both q and $q \circ p$ are cartesian fibrations. The following assertions are equivalent:

(1) The map p induces a cartesian equivalence $X^\natural \to Y^\natural$ in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(2) There exists a map $r : Y \to X$ which is a strong homotopy inverse to p, in the sense that $p \circ r$ and $r \circ p$ are both strongly homotopic to the identity.

(3) The map p induces a categorical quivalence $X_s \to Y_s$ for each vertex s of S.

Model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$

Let us now define the model structure we want on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$:
There exists a left proper combinatorial model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which can be described as follows:

(C) The cofibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the morphisms $p : X \to Y$ in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which are cofibrations when regarded as morphisms of simplicial sets.

(W) The weak equivalences in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the cartesian equivalences.

(F) The fibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the maps which have the right lifting property with respect to every map which is simultaneously a cofibration and a weak equivalence.

Model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$

Let us now define the model structure we want on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$:
There exists a left proper combinatorial model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which can be described as follows:

(C) The cofibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the morphisms $p : X \to Y$ in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which are cofibrations when regarded as morphisms of simplicial sets.

(W) The weak equivalences in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the cartesian equivalences.

(F) The fibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the maps which have the right lifting property with respect to every map which is simultaneously a cofibration and a weak equivalence.

Model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$

Let us now define the model structure we want on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$:
There exists a left proper combinatorial model structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which can be described as follows:

(C) The cofibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the morphisms $p : X \to Y$ in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ which are cofibrations when regarded as morphisms of simplicial sets.

(W) The weak equivalences in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the cartesian equivalences.

(F) The fibrations in $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ are the maps which have the right lifting property with respect to every map which is simultaneously a cofibration and a weak equivalence.

Let S be a simplicial set. Let's regard $(\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$ as a simplicial category with mapping objects given by $\mathrm{Map}^\sharp_S(X, Y)$. Then $(\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$ is a simplicial model category.

Of course, we can define a second simplicial structure on $(\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$, where the simplicial mapping spaces are given by $\mathrm{Map}^\flat_S(X, Y)$. This simplicial structure is not compatible with the cartesian model structure: for fixed $X \in (\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$ the functor $A \mapsto A^\flat \times X$ does not carry weak homotopy equivalences to cartesian equivalences. It does, however, carry categorical equivalences to cartesian equivalences, and consequently $(\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$ is endowed with the structure of a $\mathsf{Set}_{\boldsymbol\Delta}$-enriched model category, where we regard $\mathsf{Set}_{\boldsymbol\Delta}$ as equipped with the Joyal model structure. It's actually closer to the truth to say that $(\mathsf{Set}^+_{\boldsymbol\Delta})_{/S}$ is a model for an ∞-bicategory.

Let S be a simplicial set. Let's regard $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ as a simplicial category with mapping objects given by $\mathrm{Map}^\sharp_S(X, Y)$. Then $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is a simplicial model category.
Of course, we can define a second simplicial structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$, where the simplicial mapping spaces are given by $\mathrm{Map}^\flat_S(X, Y)$. This simplicial structure is not compatible with the cartesian model structure: for fixed $X \in (\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ the functor $A \mapsto A^\flat \times X$ does not carry weak homotopy equivalences to cartesian equivalences.
It does, however, carry categorical equivalences to cartesian equivalences, and consequently $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is endowed with the structure of a $\mathsf{Set}_{\mathbf{\Delta}}$-enriched model category, where we regard $\mathsf{Set}_{\mathbf{\Delta}}$ as equipped with the Joyal model structure. It's actually closer to the truth to say that $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is a model for an ∞-bicategory.

Let S be a simplicial set. Let's regard $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ as a simplicial category with mapping objects given by $\mathrm{Map}^\sharp_S(X,Y)$. Then $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is a simplicial model category.
Of course, we can define a second simplicial structure on $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$, where the simplicial mapping spaces are given by $\mathrm{Map}^\flat_S(X,Y)$. This simplicial structure is not compatible with the cartesian model structure: for fixed $X \in (\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ the functor $A \mapsto A^\flat \times X$ does not carry weak homotopy equivalences to cartesian equivalences. It does, however, carry categorical equivalences to cartesian equivalences, and consequently $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is endowed with the structure of a $\mathsf{Set}_{\mathbf{\Delta}}$-enriched model category, where we regard $\mathsf{Set}_{\mathbf{\Delta}}$as equipped with the Joyal model structure. It's actually closer to the truth to say that $(\mathsf{Set}^+_{\mathbf{\Delta}})_{/S}$ is a model for an ∞-bicategory.

Table of Contents

The different model structures

We have a plethora of model structures on categories of simplicial sets over the simplicial set S:

(0) Let $\mathscr{C}_0$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the Joyal model structure. Cofibrations are monomorphisms of simplicial sets, and weak equivalences are categorical equivalences.

(1) Let $\mathscr{C}_1$ denote $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ endowed with the marked model structure. Cofibrations are maps $(X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ which induce monomorphisms $X \to Y$, and the weak equivalences are cartesian equivalences.

The different model structures

We have a plethora of model structures on categories of simplicial sets over the simplicial set S:

(0) Let $\mathscr{C}_0$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the Joyal model structure. Cofibrations are monomorphisms of simplicial sets, and weak equivalences are categorical equivalences.

(1) Let $\mathscr{C}_1$ denote $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ endowed with the marked model structure. Cofibrations are maps $(X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ which induce monomorphisms $X \to Y$, and the weak equivalences are cartesian equivalences.

(2) Let $\mathscr{C}_2$ denote $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S}$ endowed with the following localization of the cartesian model structure: $f : (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a cofibration if $X \to Y$ is a monomorphism, and a weak equivalence if $f : X^\sharp \to Y^\sharp$ is a marked equivalence in $(\mathrm{Set}^+_{\boldsymbol{\Delta}})_{/S}$.

(3) Let $\mathscr{C}_3$ denote $(\mathrm{Set}_{\boldsymbol{\Delta}})_{/S}$ endowed with the covariant model structure. The cofibrations are monomorphisms and the weak equivalences are contravariant equivalences.

(4) Let $\mathscr{C}_4$ denote $(\mathrm{Set}_{\boldsymbol{\Delta}})_{/S}$ endowed with the usual homotopic-theoretic model structure. The cofibrations are monomorphisms of simplicial sets, and the weak equivalences are weak homotopy equivalences of simplicial sets.

(2) Let $\mathscr{C}_2$ denote $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ endowed with the following localization of the cartesian model structure: $f : (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a cofibration if $X \to Y$ is a monomorphism, and a weak equivalence if $f : X^\sharp \to Y^\sharp$ is a marked equivalence in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(3) Let $\mathscr{C}_3$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the covariant model structure. The cofibrations are monomorphisms and the weak equivalences are contravariant equivalences.

(4) Let $\mathscr{C}_4$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the usual homotopic-theoretic model structure. The cofibrations are monomorphisms of simplicial sets, and the weak equivalences are weak homotopy equivalences of simplicial sets.

(2) Let $\mathscr{C}_2$ denote $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ endowed with the following localization of the cartesian model structure: $f : (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a cofibration if $X \to Y$ is a monomorphism, and a weak equivalence if $f : X^\sharp \to Y^\sharp$ is a marked equivalence in $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(3) Let $\mathscr{C}_3$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the covariant model structure. The cofibrations are monomorphisms and the weak equivalences are contravariant equivalences.

(4) Let $\mathscr{C}_4$ denote $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ endowed with the usual homotopic-theoretic model structure. The cofibrations are monomorphisms of simplicial sets, and the weak equivalences are weak homotopy equivalences of simplicial sets.

The Quillen adjunctions

There exists a sequence of Quillen adjunctions:

$$\mathscr{C}_0 \underset{G_0}{\overset{F_0}{\rightleftarrows}} \mathscr{C}_1 \underset{G_1}{\overset{F_1}{\rightleftarrows}} \mathscr{C}_2 \underset{G_2}{\overset{F_2}{\rightleftarrows}} \mathscr{C}_3 \underset{G_3}{\overset{F_3}{\rightleftarrows}} \mathscr{C}_4$$

Description of adjunction functors

The functors may be described as follows:

(A0) G_0 is the forgetful functor from $(\mathrm{Set}_{\mathbf{\Delta}}^+)_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$, which ignores the collection of marked edges. F_0 is left adjoint to G_0, given by $X \mapsto X^\flat$. The Quillen adjunction (F_0, G_0) is a Quillen equivalance if S is a Kan complex.

(A1) F_1 and G_1 are identity functors on $(\mathrm{Set}_{\mathbf{\Delta}}^+)_{/S}$.

(A2) F_2 is the forgetful functor from $(\mathrm{Set}_{\mathbf{\Delta}}^+)_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ which ignores the collection of marked edges. G_2 is right adjoint to F_2 given by $X \mapsto X^\sharp$. The Quillen adjunction (F_2, G_2) is a Quillen equivalence for every simplicial set S.

(A3) F_3 and G_3 are identity functors on $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$. The Quillen adjunction (F_3, G_3) is Quillen equivalence whenever S is a Kan complex.

Description of adjunction functors

The functors may be described as follows:

(A0) G_0 is the forgetful functor from $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$, which ignores the collection of marked edges. F_0 is left adjoint to G_0, given by $X \mapsto X^\flat$. The Quillen adjunction (F_0, G_0) is a Quillen equivalance if S is a Kan complex.

(A1) F_1 and G_1 are identity functors on $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(A2) F_2 is the forgetful functor from $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ which ignores the collection of marked edges. G_2 is right adjoint to F_2 given by $X \mapsto X^\sharp$. The Quillen adjunction (F_2, G_2) is a Quillen equivalence for every simplicial set S.

(A3) F_3 and G_3 are identity functors on $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$. The Quillen adjunction (F_3, G_3) is Quillen equivalence whenever S is a Kan complex.

Description of adjunction functors

The functors may be described as follows:

(A0) G_0 is the forgetful functor from $(\text{Set}_{\Delta}^{+})_{/S}$ to $(\text{Set}_{\Delta})_{/S}$, which ignores the collection of marked edges. F_0 is left adjoint to G_0, given by $X \mapsto X^{\flat}$. The Quillen adjunction (F_0, G_0) is a Quillen equivalance if S is a Kan complex.

(A1) F_1 and G_1 are identity functors on $(\text{Set}_{\Delta}^{+})_{/S}$.

(A2) F_2 is the forgetful functor from $(\text{Set}_{\Delta}^{+})_{/S}$ to $(\text{Set}_{\Delta})_{/S}$ which ignores the collection of marked edges. G_2 is right adjoint to F_2 given by $X \mapsto X^{\sharp}$. The Quillen adjunction (F_2, G_2) is a Quillen equivalence for every simplicial set S.

(A3) F_3 and G_3 are identity functors on $(\text{Set}_{\Delta})_{/S}$. The Quillen adjunction (F_3, G_3) is Quillen equivalence whenever S is a Kan complex.

Description of adjunction functors

The functors may be described as follows:

(A0) G_0 is the forgetful functor from $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$, which ignores the collection of marked edges. F_0 is left adjoint to G_0, given by $X \mapsto X^\flat$. The Quillen adjunction (F_0, G_0) is a Quillen equivalance if S is a Kan complex.

(A1) F_1 and G_1 are identity functors on $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$.

(A2) F_2 is the forgetful functor from $(\mathrm{Set}^+_{\mathbf{\Delta}})_{/S}$ to $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$ which ignores the collection of marked edges. G_2 is right adjoint to F_2 given by $X \mapsto X^\sharp$. The Quillen adjunction (F_2, G_2) is a Quillen equivalence for every simplicial set S.

(A3) F_3 and G_3 are identity functors on $(\mathrm{Set}_{\mathbf{\Delta}})_{/S}$. The Quillen adjunction (F_3, G_3) is Quillen equivalence whenever S is a Kan complex.