Ex Show that the functor which sends $\mathcal{W}$ to isomorphisms, $\mathrm{Top} \longrightarrow \mathrm{HoTop}$, satisfies the following commutative diagram, for any category $\mathcal{C}$ —

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, "P"] \arrow[dr, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowright", shift left=4ex] & \mathrm{HoTop} \arrow[d, dashed, "\exists!\,\overline{\mathcal{F}}"] \\
 & \mathcal{C}
\end{tikzcd}$$

Proof To localize Top with respect to $\mathcal{W}$, let us use the Gabriel – Zisman category of fractions. $\mathrm{Top}[W^{-1}] = \mathrm{Ho\,Top}$, and it has the same objects as Top, and morphisms are modulo the following equiv. relations —

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \approx \cdot \xrightarrow{gf} \cdot$$

$$\left.\begin{array}{c} \overset{\cdot}{\circlearrowleft} \\ \cdot \xrightarrow{s} \cdot \xleftarrow{s} \cdot \\ \cdot \xleftarrow[t]{} \cdot \xrightarrow[t]{} \cdot \end{array}\right\} \text{may be removed}$$

$\mathrm{Top} \longrightarrow \mathrm{Ho\,Top}$ is hence an epimorphism.

A homotopy in Top between spaces X and Y is the map $\mathcal{H} : X \times I \longrightarrow Y$ that satisfies

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & X \times I \arrow[d, "\mathcal{H}"] & X \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

such that $\left.\begin{aligned} \mathcal{H}(-,0) &= f \\ \mathcal{H}(-,1) &= g \end{aligned}\right\}$

Now, applying functor $\mathcal{F}$ —

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, hook, "\mathcal{F}(i_0)"] \arrow[dr, "\mathcal{F}(f)"'] & \mathcal{F}(X \times I) \arrow[d, "\tilde{\mathcal{H}}"] & \mathcal{F}(X) \arrow[l, hook', "\mathcal{F}(i_1)"'] \arrow[dl, "\mathcal{F}(g)"] \\
& \mathcal{F}(Y) &
\end{tikzcd}$$

is a homotopy in $\mathcal{C}$.

Now consider $r: X \times I \longrightarrow X$

defined by $(x, -) \longmapsto x$.

Now, r is a retract of

$i_1 : X \longrightarrow X \times I$, so

$r i_1 \cong id_X \Rightarrow i_1 = r^{-1}$.

Now, $\mathcal{F}(f) = \mathcal{F}(i_0)\,\mathcal{F}(r)\,\mathcal{F}(g)$

If $\mathcal{F}(i_1) = \mathcal{F}(r)^{-1}$, then this becomes

$$\mathcal{F}(f) = \mathcal{F}(i_0)\,\mathcal{F}(i_1)^{-1}\,\mathcal{F}(g)$$

$$\begin{tikzcd} X \arrow[r, hook, "\mathcal{F}(i_0)"] & X \times I \arrow[r, "\mathcal{F}(i_1)^{-1}"] & X \end{tikzcd}$$

Now, $i_0 : (x, 1) \longmapsto x$

$i_1^{-1} : (x, -) \longmapsto x$

Clearly, if $\mathcal{F}(i_1)$ were invertible, $\mathcal{F}(i_1)^{-1}$ is a retract of $\mathcal{F}(i_0)$ and

$$\mathcal{F}(i_0)\,\mathcal{F}(i_1)^{-1}\,\mathcal{F}(g) = \mathcal{F}(g)$$
$$= \mathcal{F}(f)$$

The condition on $\mathcal{F}(i_1)$ being invertible can then be stated as the condition that $\mathcal{F}(i_1)$ is an isomorphism. It remains to show that the invertibility of $\mathcal{F}(i_1)$ leads to the definition of HoTop.

$$\begin{tikzcd}
X \arrow[r, hook, "i_0"] & X \times I \arrow[r, "r"] & X \\
X \arrow[ur, hook, "i_1"'] & &
\end{tikzcd}$$

In HoTop, identity arrows may be removed, hence leading to the removal of $ri_1 \simeq id_X$ Hence $F(i_1)$ can be written as $F(r)^{-1}$ in HoTop, leading to the existence of the functor $\tilde{F}$.

Ex Show that

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "f_1"] & \mathrm{cone}(f) \arrow[d, "f_2"] \\
 & \mathrm{cone}(f_2) & \mathrm{cone}(f_1) \arrow[l, "f_3"']
\end{tikzcd}$$

is h-coexact.

Proof Cone(f) is defined by the pushout —

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{Cone}(X) \arrow[d] \\
Y \arrow[r, "f_1"] & \mathrm{Cone}(f)
\end{tikzcd}$$

$$\mathrm{Cone}(f) \equiv Y \sqcup_f \mathrm{Cone}(X) = \frac{M_f}{j(X)}$$

where

$Mf = Y \sqcup_f (X \times I)$

$j : X \longrightarrow Mf$ sends $x \mapsto (x, 1)$

j is obviously a cofibration, and f can be factored as:

$f : X \xrightarrow{j} Mf \xrightarrow{r} Y$

where r is a retract.

Cf is obtained by applying j and taking the associated quotient space.

Hence, we have an inclusion $i : Y \hookrightarrow Cf$

The inclusion $Y \hookrightarrow Cf$ is obviously a cofibration since it is obtained as the pushout of the cofibration $X \rightarrowtail CX$ and the map $f : X \longrightarrow Y$.

$X \rightarrowtail CX$ sends $x \longmapsto (x, 0)$ and $X \simeq CX$

— hence $Y \rightarrowtail Cf$ is a cofibration.

By the definition of h-coexactness, it only remains to show that the following sequence of pointed sets is exact, since the pattern generalizes —

$$[Cf, Z] \longleftarrow [Y, Z] \longleftarrow [X, Z]$$

for any pointed space Z. To see this, consider the commutative diagram —

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\simeq"'] \arrow[dr, "h"] & Y \arrow[r, tail, "i"] \arrow[d, "g"] & Cf \arrow[dl, dashed, "\exists!(g \sqcup h)"] \\
CX \arrow[r, "h"'] & Z &
\end{tikzcd}$$

h is obtained as the composite $g \circ f$, which can either be viewed as a map $X \longrightarrow Z$ or a map $CX \longrightarrow Z$, since $X \overset{\sim}{\rightarrowtail} CX$ is a cofibration which is a homotopy equivalence.

Now, since $Cf \equiv CX \sqcup_f Y$, and h can be viewed as $h : CX \longrightarrow Z$, we prove the existence of the dotted arrow, which is obtained as the pushout of maps g and h.

Ex Show that all the bottom maps of the following commutative diagram are homeomorphic to ΣX

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_1"] \arrow[dr, phantom, "\ulcorner", very near end] & Y \arrow[r, "i_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\ulcorner", very near end] & \mathrm{Cone}(Y) \arrow[d, "j_1"] \\
CX \arrow[r, "j"] \arrow[d, "p"] & Cf \arrow[r, "f_2"] \arrow[d, "p(f)"] & Cf_1 \arrow[d, "q(f)", "\sim"'] \\
CX/i_1(X) \arrow[r, "\cong"] \arrow[d, equal] & Cf/f_1(Y) \arrow[r, "\cong"] \arrow[d, equal] & Cf_1/j_1 CY \arrow[d, equal] \\
\Sigma X & \Sigma X & \Sigma X
\end{tikzcd}$$

$f : X \longrightarrow Y$ can be factored as $f : X \overset{i}{\rightarrowtail} Mf \xrightarrow{\sim} Y$ as in the previous exercise, and

$X \overset{\sim}{\rightarrowtail} CX$ is a cofibration which is a homotopy equivalence that sends $x \longmapsto (x, 0)$.

Now, $\Sigma X \equiv X \wedge S^1 = X \wedge \frac{I}{\partial I}$

$CX \equiv X \wedge I$

$$\begin{aligned} i_1 : X &\longrightarrow CX \\ x &\longmapsto (x, 0) \end{aligned}$$

when CX is quotiented

by $i_1(X)$, we obtain ΣX.

Next, consider

$$Cf \equiv Y \sqcup_f CX$$

and the inclusion $f_1 : Y \hookrightarrow Cf$. This is an inclusion because r is a retract, which admits a section $Y \hookrightarrow Mf$, and Cf is a quotient of Mf by the cofibration $X \rightarrowtail Mf$.

Let us now inspect Cf and f_1 —

$$Cf \equiv Y \sqcup_f CX$$

$$f_1 : Y \hookrightarrow Y \sqcup_f CX$$

Since $X \overset{\sim}{\rightarrowtail} CX$ is a cofibration, and $f : X \to Y$ is any map, f_1 is a cofibration obtained as the pushout of a cofibration with f.

f_1 sends $y \longmapsto y$ in Y

and $(x, t) \longmapsto (x, 0)$ in CX

Hence, quotienting the space $Y \sqcup_f CX$ by the cofibration $Y \rightarrowtail Y \sqcup_f CX$ yields

$$\frac{CX}{x \mapsto (x,0)} = \Sigma X.$$

Next, observe that Cf_1 is obtained as the pushout of the cofibration $Y \rightarrowtail Cf$ and the inclusion $Y \hookrightarrow CY$.

Hence, $j_1 : CY \rightarrowtail Cf_1$ is a cofibration as well.

$$j_1 : CY \rightarrowtail Cf_1$$

$$: CY \rightarrowtail Cf \sqcup_{f_1} CY$$

where $Cf \equiv Y \sqcup_f CX$

$$j_1 : CY \rightarrowtail CY \sqcup_{f_1} Y \sqcup_f CX$$

$Cf_1 / j_1 CY$ remains to be inspected.

j_1 sends $((y, s), t) \mapsto (y, s)$ in CY

$(y, t) \mapsto y$ at 0 in Y

$((x, s), t) \mapsto (x, s)$ in CX

Hence, the quotient is identical to ΣX.

Ex Show that $q(f)$, from the previous exercise, is a homotopy equivalence.

Proof $q(f) : Cf_1 \longrightarrow Cf_1/j_1CY$

We have already shown that j_1 is a cofibration, and it remains to prove the general lemma that given a cofibration $i : A \rightarrowtail X$, the following map is a homotopy equivalence

$\psi : Ci \longrightarrow Ci/CA \cong X/A$

Now, since i is a cofibration, there exists $r : X \wedge I_+ \longrightarrow Mi \equiv X \sqcup_i (A \wedge I_+)$ In r, collapse $X \times \{1\}$ in the source, and $A \times \{1\}$ in the target. The composite yields a map $\phi : X \longrightarrow Ci$ The map r collapses A to $\{*\}$, and hence induces the map $\mu : X/A \longrightarrow Ci$. Gluing μ with ψ yields a map $Ci \longrightarrow Ci$ such that $\mu \cdot \psi \simeq id$. Now, r restricted to the.

space $A \wedge I_+$ is the identity, and this glues together with the map $CA \wedge I_+ \longrightarrow CA$ given by

$\xi : ((x, s), t) \longmapsto (x, \max(s,t))$

to yield a homotopy $Ci \wedge I_+ \longrightarrow Ci$, finally giving $\psi . \mu \simeq id$.

Ex We use $\tau : \Sigma X \longrightarrow \Sigma X$ given by $(x, t) \longmapsto (x, 1-t)$ to denote the orientation reversing homotopy of Σ.

In the following diagram, show that the left and right triangles are commutative, and that the middle triangle is homotopy commutative

$$\begin{tikzcd}
Cf \arrow[r, "f_2"] \arrow[dr, "p(f_1)"'] & Cf_1 \arrow[r, "f_3"] \arrow[d, "q(f)"] \arrow[dr, "p(f_2)"] & Cf_2 \arrow[d, "q(f_1)"] \\
 & \Sigma X \arrow[r, "\Sigma f . \tau"] & \Sigma Y
\end{tikzcd}$$

<u>Proof</u>

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\sim"] & Y \arrow[r, tail, "\sim"] \arrow[d, tail, "f_1"] & CY \arrow[d, "f_2"] \\
CX \arrow[r] & Cf \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Cf_1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Observe that Cf_1 is obtained by gluing the bases of CX and CY along the map $f : X \longrightarrow Y$ collapsing out CY from Cf_1 is equivalent to collapsing out Y from Cf — hence, the left triangle commutes. A homotopy h from $Cf_1 \wedge I_+ \longrightarrow \Sigma Y$ from $p(f_2)$ to $\Sigma f \cdot \tau \cdot q(f)$ is given by —

$$h : (CY \sqcup_{f_1} CX \sqcup_f Y) \wedge I_+$$

$$\longrightarrow Y \wedge S^1 \equiv \Sigma Y$$

$$(y, t) \longmapsto (y, t) \text{ in } Y$$

$$((y, s), t) \longmapsto (y, t + s - st) \text{ in } CY$$

$$((x, s), t) \longmapsto (f(x), t - st) \text{ in } CX$$

Finally, the right triangle is commutative, because collapsing out CY from Cf_1 is equivalent to collapsing out Cf from Cf_1.

<u>Ex</u> From the previous exercises, conclude that

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "f_1"] & Cf \arrow[r, "p(f_1)"] & \Sigma X \arrow[d, "\Sigma f"'] \\
 & & & \Sigma Y
\end{tikzcd}$$

is h-coexact.

<u>Proof</u> We have already shown

(i) $X \xrightarrow{f} Y \xrightarrow{f_1} Cf \xrightarrow{f_2} Cf_1 \xrightarrow{f_3} Cf_2$

is h-coexact.

(ii) The triangles

$$\begin{tikzcd}
Cf \arrow[r] \arrow[dr] & Cf_1 \arrow[d] \\
& \Sigma X
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
Cf_1 \arrow[r] \arrow[dr] & Cf_2 \arrow[d] \\
& \Sigma Y
\end{tikzcd}$$

commute, and glue together with a homotopy.

Hence, these two results imply that

$$X \longrightarrow Y \longrightarrow Cf \longrightarrow \Sigma X \longrightarrow \Sigma Y$$

is h-coexact.

<u>Ex</u> Show that $\exists$ a homeomorphism $\chi : C\Sigma X \longrightarrow \Sigma C X$

such that $X \circ \Sigma f_1 = \Sigma f_1$

<u>Proof</u>

$$\begin{tikzcd}
 & & CCf \arrow[r, tail, "\sim"] & \Sigma Cf \\
X \arrow[r, "f"] \arrow[d, tail, "\sim"] & Y \arrow[r, tail, "f^1"] \arrow[d, tail, "f_1"] & Cf \arrow[r, tail, "\sim"] \arrow[u, tail, "f_4", "\sim"'] \arrow[d, tail, "f_3"', "\sim"] \arrow[ur, phantom, "P"] \arrow[dr, phantom, "Q"] & CCf \arrow[u, tail, "\sim"] \arrow[d, tail, "\sim"] \arrow[ul, phantom, "\lrcorner", very near start] \\
CX \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner", very near start] & Cf \arrow[r, tail] \arrow[d, tail, "\sim"] \arrow[ur, phantom, "\ulcorner", very near start] & \Sigma f \arrow[r, tail, "\sim"] \arrow[d, tail, "\sim"] \arrow[ur, phantom, "\ulcorner", very near start] \arrow[dr, phantom, "R"] & C\Sigma f \arrow[d, tail, "\sim"] \\
CCX \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & CCf \arrow[r, tail, "\sim"] \arrow[ur, phantom, "\ulcorner", very near start] & C\Sigma f \arrow[r, tail, "\sim"] \arrow[ur, phantom, "\ulcorner", very near start] & \Sigma\Sigma f
\end{tikzcd}$$

Square Q is necessarily a pushout. Next, notice that the two homotopy equivalences in square P compose to yield $\Sigma Cf \simeq Cf$.

Similarly, in square R, we get $\Sigma f \simeq \Sigma \Sigma f$.

Finally, viewing squares Q and R together, we get the result that the two vertical arrows of Q are also homotopy equivalences.

Together, this yields $\overline{\Sigma} Cf \simeq \Sigma Cf$.

Moreover, it is a bijection of sets as

$\Sigma C f = C C f \sqcup_{f_4} C C f$

and $C \Sigma f = C C f \sqcup_{f_3} \Sigma f$

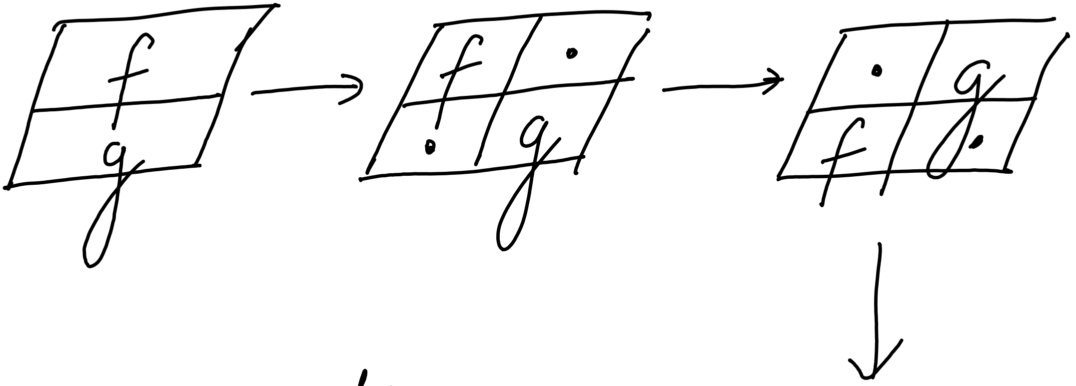
χ can equivalently be written as a map $C C f \xrightarrow{\cong} \Sigma f$, and since f is arbitrary, a map $C C f_1 \xrightarrow{\cong} \Sigma f_1$.

$\chi_{\circ} \Sigma f_1 : C C \Sigma f_1 \longrightarrow \Sigma \Sigma f_1$

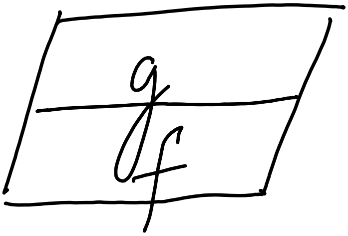
yielding the required equality.

Ex Show that $+_1$, the product operation on $[\Sigma^2 X, Y]$ is abelian.

Proof Let $f, g : [\Sigma^2 X, Y]$. By definition of Σ, $f, g : S^2 \longrightarrow \mathcal{F}(X, Y)$, where $\mathcal{F}$ is the function space. Further, since $S^2 \sim I^2/\partial I^2$, the homotopy between $f+g$ and $g+f$ can be pictured diagramatically as —



Hence, the product operation on



$[\Sigma^2 X, Y]$ is abelian.

Ex Describe $\eta : X \longrightarrow \Omega\Sigma X$ and $\varepsilon : \Sigma\Omega X \longrightarrow X$, the unit and counit of the $\Sigma - \Omega$ adjunction.

Proof $\Sigma : \mathrm{Top}_* \underset{\perp}{\rightleftarrows} \mathrm{Top}_* : \Omega$

For based spaces X and Y,

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

or $F(\Sigma X, Y) \cong F(X, \Omega Y)$

where $\Sigma X = X \wedge S^1$

and $\Omega X = F(S^1, X)$

$\eta : X \longrightarrow \Omega \Sigma X$ is given by

$$\eta : X \longrightarrow F(S^1, X \wedge S^1)$$

and $\varepsilon : \Sigma \Omega X \longrightarrow X$

is given by

$$\varepsilon : F(S^1, X) \wedge S^1 \longrightarrow X$$

__Ex__ If $i : A \longrightarrow X$ is a cofibration, and $f : A \to B$ is any map, then the induced map $g : B \longrightarrow B \sqcup_f X$ is a cofibration.

__Proof__

$$\begin{tikzcd}
A \arrow[rrrr] \arrow[dr, "f"] \arrow[dddd, "i"'] & & & & A \times I \arrow[dl] \arrow[dddd] \\
& B \arrow[rr] \arrow[dd, "g"'] & & B \times I \arrow[dl] \arrow[dd] & \\
& & Y & & \\
& B \sqcup_f X \arrow[ur, "h"] \arrow[rr] & & (B \sqcup_f X) \times I \arrow[ul, dashed, "\exists ! j"'] & \\
X \arrow[ur, "\tilde{h}"] \arrow[rrrr] & & & & X \times I \arrow[ul, "\tilde{j}"']
\end{tikzcd}$$

$h\tilde{h} : X \longrightarrow Y$ determines a homotopy from X to Y.

Since $i : A \longrightarrow X$ is a cofibration, there exists a unique $j \, \tilde{j} : X \times I \longrightarrow Y$, and this map factors uniquely through $\tilde{j} : X \longrightarrow (B \sqcup_f X) \times I$, yielding a unique map $j : (B \sqcup_f X) \times I \longrightarrow Y$. The latter map realizes $g : B \longrightarrow B \sqcup_f X$ as a cofibration.

Ex Show that any map $f: X \longrightarrow Y$ can be factored as a cofibration followed by a homotopy equivalence: f is equivalent to

$$\begin{tikzcd} X \arrow[r, "i"] & cyl(f) \arrow[r, "r", "\sim"'] & Y \arrow[l, bend left, "s"] \end{tikzcd}$$

Proof $Cyl(f)$ is given by $(X \times I) \sqcup_f Y$. The map $i: X \longrightarrow cyl(f)$ is given by

$i(x) = (x, 1)$

In other words, it is an embedding of X in $\mathrm{Cyl}(f)$ at $t = 1$.

$r : \mathrm{Cyl}(f) \longrightarrow Y$ is $(X \times I) \sqcup_f Y \longrightarrow Y$. It is given by $r(x, s) = f(x)$ at X and $r(y) = y$.

Furthermore, there exists a map $s : Y \longrightarrow \mathrm{Cyl}(f)$ such that $s \circ r \sim \mathrm{id}_{\mathrm{Cyl}(f)}$ and $r \circ s = \mathrm{id}_Y$

where $id_{Cyl(f)}$ can be defined via a homotopy

$$Cyl(f) \times I \xrightarrow{q} Cyl(f)$$

given by:

$$q((x,t),s) = (x, t(1-s)).$$

Hence, r is a retraction, and s is its section. Together they define a homotopy equivalence between $Cyl(f)$ and Y. i is obviously a cofibration, being an inclusion with a closed image. We can also directly check that i satisfies HEP —

$$\begin{tikzcd}
X \arrow[rr, "i_0"] \arrow[dr, "f"] \arrow[dd, "i"'] & & X \times I \arrow[dl, "h"'] \arrow[dd, "j"] \\
& Y \arrow[dl, shift left, "r"] & \\
\mathrm{Cyl}(f) \arrow[ur, shift left, "s"] \arrow[rr, "i_1"] & & \mathrm{Cyl}(f) \times I \arrow[ul, dashed, "\exists ! \tilde{h}"'] \arrow[ll, bend left, "q"]
\end{tikzcd}$$

We have shown the deformation $\mathrm{Cyl}(f) \times I \longrightarrow \mathrm{Cyl}(f)$ and the section

$Cyl(f) \longrightarrow Y$. Hence, these two maps compose as $sq = \tilde{h}$ yielding a map $Cyl(f) \times I \longrightarrow Y$. The commutativity of the diagram bears witness to the fact that $i : X \longrightarrow Cyl(f)$ is a cofibration.

Ex If $f : A \rightarrowtail B$ and $g : B \rightarrowtail C$ are two cofibrations, show that $gf : A \rightarrowtail C$ is a cofibration.

Proof

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, tail, "f"'] & & A \times I \arrow[dl, "n"'] \arrow[dd, "j"] \\
& Y & \\
B \arrow[ur, "l"] \arrow[rr, "i_1"] \arrow[dd, tail, "g"'] & & B \times I \arrow[ul, dashed, "h"'] \arrow[dl, "p"'] \arrow[dd, "k"] \\
& Z & \\
C \arrow[ur, "m"] \arrow[rr, "i_2"] & & C \times I \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, tail, "gf"'] & & A\times I \arrow[dl, "pj"'] \arrow[dd, "kj"] \\
& Z & \\
C \arrow[ur, "m"] \arrow[rr, "i_2"'] & & C\times I \arrow[ul, dashed, "\tilde{h}"']
\end{tikzcd}$$

The commutativity of the above diagram witnesses the fact that gf is a cofibration.

Ex Show that cofibrations are stable under coproduct. That is, given cofibrations $i: A \rightarrowtail X$ and $j: B \rightarrowtail Y$, show that $i \sqcup j : A \sqcup B \rightarrowtail X \sqcup Y$ is a cofibration

$$\begin{tikzcd}
A \arrow[rr, "m"] \arrow[dd, tail, "i"'] & & A\times I \arrow[dl] \arrow[dd, "k"] \\
& Cyl(i) & \\
X \arrow[ur] \arrow[rr, "p"'] & & X\times I \arrow[ul, dashed, "h"']
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[rr, "n"] \arrow[dd, tail, "j"'] & & B\times I \arrow[dl] \arrow[dd, "l"] \\
& Cyl(j) & \\
Y \arrow[ur] \arrow[rr, "q"'] & & Y\times I \arrow[ul, dashed, "h_2"']
\end{tikzcd}$$

$$\begin{tikzcd}
A\sqcup B \arrow[rr, "m\sqcup n"] \arrow[dd, "i\sqcup j"'] & & (A\sqcup B)\times I \arrow[dl] \arrow[dd, "k\sqcup l"] \\
& Cyl(i\sqcup j) & \\
X\sqcup Y \arrow[ur] \arrow[rr, "p\sqcup q"'] & & (X\sqcup Y)\times I \arrow[ul, dashed, "h"']
\end{tikzcd}$$

Notice first that $(A\sqcup B)\times I$

$=(A\times I)\sqcup(B\times I)$ and

$(X\sqcup Y)\times I=(X\times I)\sqcup(Y\times I)$

Hence, a map $(A\sqcup B)\times I\longrightarrow$

$(X\sqcup Y)\times I$ is equal to $k\sqcup l$

Notice next that

$$Cyl(i \sqcup j) =$$

$$((A \sqcup_i B) \times I) \sqcup_{i \sqcup j} (X \sqcup_j Y)$$

$$\begin{tikzcd}
\cdot \arrow[r, "m \sqcup n"] \arrow[d, "i \sqcup j"'] & (A \times I) \sqcup (B \times I) \arrow[d] \\
X \sqcup Y \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & Cyl(i \sqcup j)
\end{tikzcd}$$

This cylinder object can be rewritten as

$$(A \times I \sqcup_i X) \sqcup (B \times I \sqcup_j Y)$$

$= Cyl(i) \sqcup Cyl(j)$

Hence, the following diagram yields the required result —

$$\begin{tikzcd}
A \sqcup B \arrow[rr, "m \sqcup n"] \arrow[dd, tail, "i \sqcup j"'] & & (A \times I) \sqcup (B \times I) \arrow[dl] \arrow[dd, "k \sqcup l"] \\
& Cyl(i) \sqcup Cyl(j) & \\
X \sqcup Y \arrow[ur] \arrow[rr, "p \sqcup q"'] & & (X \times I) \sqcup (Y \times I) \arrow[ul, dashed, "\exists!"', "h_1 \sqcup h_2 = \tilde{h}"]
\end{tikzcd}$$

$\tilde{h} = h_1 \sqcup h_2$ witnesses the fact that $i \sqcup j$ is a cofibration.

Ex Show that, if $f : C \twoheadrightarrow D$ and $g : D \twoheadrightarrow E$ are fibrations, then their composite $gf : C \twoheadrightarrow E$ is a fibration!

Proof

$$\begin{tikzcd}
C \arrow[dd, two heads, "f"'] & & C^I \arrow[ll, "i_0"'] \arrow[dd, "f^I"] \\
& X \arrow[ul, "p"] \arrow[ur, dashed, "h"'] \arrow[dr, "q"'] & \\
D \arrow[dd, two heads, "g"'] & & D^I \arrow[ll, "i_1"'] \arrow[dd, "g^I"] \\
& Y \arrow[ul, "r"] \arrow[ur, dashed, "\tilde{h}"'] \arrow[dr, "s"'] & \\
E & & E^I \arrow[ll, "i_2"]
\end{tikzcd}$$

$$\begin{tikzcd}
C \arrow[dd, two heads, "gf"'] & & C^I \arrow[ll, "i_0"'] \arrow[dd, "g^I f^I"] \\
& X \arrow[ul, "p"] \arrow[ur, dashed, "h"] \arrow[dr, "g^I q"] & \\
E & & E^I \arrow[ll, "i_2"]
\end{tikzcd}$$

This commutative diagram

witnesses the fact that gf is a fibration.

Ex Show that fibrations are stable under pullback. That is, given fibration $f : E \twoheadrightarrow B$ and any map $g : A \longrightarrow B$, show that $A \times_f E \twoheadrightarrow A$ is a fibration.

$$\begin{tikzcd}
E \arrow[dddd, two heads, "f"'] & & & & E^I \arrow[llll] \arrow[dddd] \\
& A\times_f E \arrow[ul] \arrow[dd, two heads, "p"'] & & (A\times_f E)^I \arrow[ll] \arrow[ur, "h"] \arrow[dd] & \\
& & X \arrow[ul] \arrow[dr] \arrow[ur, dashed, "\tilde{h}"] & & \\
& A \arrow[dl, "g"] & & A^I \arrow[ll] \arrow[dr] & \\
B & & & & B^I \arrow[llll]
\end{tikzcd}$$

Since $h:(A\times_f E)^I \longrightarrow E^I$ is a well-defined map, $h\tilde{h}$ witnesses the fact that $f:E \twoheadrightarrow B$ is a fibration. $h\tilde{h}$ factors uniquely through h, yielding $\tilde{h}$, which witnesses the fact that $p : A \times_f E \twoheadrightarrow A$ is a fibration.

Ex Show that fibrations are stable under product. That is, given fibrations $f : E \twoheadrightarrow B$ and $g : C \twoheadrightarrow D$, show that $f \times g : E \times C \twoheadrightarrow B \times D$ is a fibration.

Proof The universal test space for fibration f is $\mathrm{Path}(f) = B \times_f E^I$ and for g is $\mathrm{Path}(g) = D \times_g C^I$

The following diagram finishes off the proof, noting that Path$(f\times g)$

$$= D \times B \times_{f\times g} E^I \times C^I$$

$$= (B \times_f E^I) \times (D \times_g C^I)$$

$$= \mathrm{Path}(f) \times \mathrm{Path}(g).$$

$$\begin{tikzcd}
E\times C \arrow[dd, two heads, "f\times g"'] & & E^I\times C^I \arrow[ll] \arrow[dd] \\
& \mathrm{Path}(f\times g) \arrow[ul] \arrow[ur, dashed, "\tilde{h}"'] \arrow[dr] & \\
B\times D & & B^I\times D^I \arrow[ll]
\end{tikzcd}$$

$\tilde{h} : \mathrm{Path}(f \times g) \longrightarrow E^I \times C^I$ is equivalent to the product of

$$\left.\begin{array}{l} h_1 : \mathrm{Path}(f) \longrightarrow E^I \\ h_2 : \mathrm{Path}(g) \longrightarrow C^I \end{array}\right\}$$

$\tilde{h} = h_1 \times h_2$ hence witnesses the fact that $f \times g$ is a fibration.

Ex Show that the geometric realization functor is left adjoint to the singular complex functor.

Proof Given simplicial set X,

$$|X| := \operatorname{colim}_{\substack{\Delta^n \to X \\ \Delta^n \downarrow X}} |\Delta^n|$$

given topological space Y, its singular complex is

$$SY := \operatorname{Hom}_{Top}(|\Delta^n|, Y)$$

Hence,

$$\operatorname{Hom}(|X|, Y) = \lim_{\Delta^n \to X} \operatorname{Hom}(|\Delta^n|, Y)$$

$$= \lim_{\Delta^n \to X} \mathrm{Hom}_{Set_{\Delta}}(\Delta^n, SY)$$

$$\cong \mathrm{Hom}_{Set_{\Delta}}(X, SY)$$

Since $X \cong \operatorname*{colim}_{\substack{\Delta^n \to X \\ \Delta^n \downarrow X}} \Delta^n$

and since Hom is left exact —

$\mathrm{Hom}(X, \lim Y_\bullet) \cong$

$\lim \mathrm{Hom}(X, Y_\bullet)$

and $\mathrm{Hom}(\operatorname{colim} X_\bullet, Y) \cong$

$\lim \mathrm{Hom}(X_\bullet, Y)$.

Ex Show that the n-skeleton of a simplicial set can be written as —

$$\begin{tikzcd}
\coprod_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}X \arrow[d, hook] \\
\coprod_{x \in NX_n} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & sk_n X
\end{tikzcd}$$

Proof The n^{th} skeleton of a simplicial set is defined as

$$(sq_n X)_m := \{x \in X_m \mid \exists k < n, \exists \varphi : [m] \twoheadrightarrow [k] \text{ surj \& non-decr.}, \exists y \in X_k \mid x = X(\varphi^{op})(y)\}$$

Proof The def. says that $(sq_n X)_m$ is an m-simplex that relates to k-simplices via X as $\varphi : [m] \twoheadrightarrow [k]$ non-decreasing, via

$$x = X(\varphi^{op})(y)$$

non-degenerate k-simplex

degenerate m-simplex

map $[m] \twoheadrightarrow [k]$

$$\mathrm{Set}_\Delta := \Delta^{op} \longrightarrow \mathrm{Set}$$

Here, $\left.\begin{matrix} k<n \\ m<n \end{matrix}\right\}$. (ref: Eilenberg-Zilber lemma)

$(\mathrm{sq}_n X)$ is hence a collection of m-simplices for $m<n$, and $X = \bigcup \mathrm{sq}_n X$.

$sk_n X$ can be thought of as a simplicial set and

$$\left.\begin{array}{l} sk_n \Delta^n = \Delta^n. \\ sk_{n-1} \Delta^n = \partial \Delta^n. \end{array}\right\}$$

Now, sk_n can be constructed using the formalization of cell-complexes. Recall that a CW-complex can be constructed as —

$$\begin{tikzcd}
J_n \times \partial D^n \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near end] & X^{(n-1)} \arrow[d] \\
J_n \times D_n \arrow[r] & X^{(n)}
\end{tikzcd}$$

In the above diagram, replacing disks by standard n-simplices yields the required result.

Note that we have used the fact that

$$X = \bigcup \mathrm{sq}_n X = \operatorname*{colim}_{\substack{\Delta^n \to X \\ \Delta^n \downarrow X}} \Delta^n$$

$$\begin{tikzcd}
\coprod_{x \in NX_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sq}_{n-1} X \arrow[d, hook] \\
\coprod_{x \in NX_n} \Delta^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{sq}_n X
\end{tikzcd}$$

where $NX_n \subset X_n$ is the set of non-degenerate simplices of degree n.

Ex Prove that Set_{Δ^1} admits all limits and colimits.

Proof $\mathrm{Set}_{\Delta^1} := \mathrm{Set}^{\Delta^{1\,op}}$
This is a functor category with functors $\Delta^{1\,op} \longrightarrow \mathrm{Set}$.
Notice that the pointwise (co)lim, (co) lim $\Delta^{1\,op} \longrightarrow \mathrm{Set}$ has codomain Set, which admits all limits and colimits. Hence, Set_{Δ^1} admits all limits and colimits.

Ex Show that $\partial\Delta^n$ can be written as —

$$\begin{tikzcd} \coprod\limits_{0\leq i<j\leq n} \Delta^{n-2} \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{i=0}^{n} \Delta^{n-1} \arrow[r] & \partial\Delta^n \end{tikzcd}$$

given by $d_j d_i = d_i d_{j-1}$ if $i<j$.

Proof $\partial\Delta^n$ is the smallest subcomplex of Δ^n containing faces $d_i(\iota_n)$ for standard n-simplex ι_n and $0 \leq i \leq n$.

$$\partial\Delta^n_j = \begin{cases} \Delta^n_j, & 0 \leq j < n \\ \text{iterated degeneracies of } \Delta^n_k, \ 0 \leq k < n, & j > n-1 \end{cases}$$

Now, consider the comma category $\Delta \downarrow \Delta^n$ —

$$\begin{tikzcd} \Delta^p \arrow[d] \arrow[dr] & \\ & \Delta^n \\ \Delta^q \arrow[ur] & \end{tikzcd}$$

$(p \leq q \leq n)$

So, $\Delta^n = \operatorname*{colim}_{\substack{\Delta^n \to \Delta^n \\ \text{in } \Delta \downarrow \Delta^n}} \Delta^n$

$\Delta^n \longrightarrow \Delta^n$ is the terminal object of $\Delta \downarrow \Delta^n$.

Now, we know that $\partial\Delta^n \hookrightarrow \Delta^n$ is an injection and we can draw the limiting cocone as —

$$\begin{tikzcd}
\Delta^p \arrow[d] \arrow[dr] \arrow[drr] & & \\
& \partial\Delta^n \arrow[r, hook, "\exists!"] & \Delta^n \\
\Delta^q \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

From the def. of $\partial\Delta^n$, we can infer p and q as the injections

$$\Delta^{n-2} \hookrightarrow \Delta^{n-1} \hookrightarrow \partial\Delta^n$$

hold.

Hence, $\partial\Delta^n = \underset{\substack{\Delta^{n-2} \hookrightarrow \Delta^{n-1} \\ \text{in } \Delta\downarrow\partial\Delta^n}}{\operatorname{colim}} \Delta^n$

As colimits can be alternatively expressed in terms of coproducts and coequilizers, this yields the required result.

Ex Consider the functor $cst : Set \longrightarrow Set_{\Delta}$ which assigns the constant simplicial set $X_n := X$, $d_i = id$, $s_i = id$ to any set X. Show that this functor is full, faithful, and representable.

Proof $X_1 = X_2 \Rightarrow cst(X_1) = cst(X_2)$

and $cst(X_1) = cst(X_2)$
$\Rightarrow X_1 = X_2$

This is easily seen, as the simplicial set corresponding to X_1 and X_2, $X_1 \neq X_2$ are unique and have a unique forgetful functor that simply picks out any X_m to yield X_1 or X_2

$$\begin{tikzcd}
X_1 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_1 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_1 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_1 \\
X_2 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_2 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_2 \arrow[r, shift left=2, "id"] \arrow[r, phantom, "\vdots"] \arrow[r, leftarrow, shift right=2] & X_2
\end{tikzcd}$$

The forgetful functor to Set (ie. $\mathrm{Set}^{\Delta^{op}} \longrightarrow \mathrm{Set}$)

yields the set we started out with, and hence cst is representable.

Ex Describe the left and right adjoints of $\mathrm{cst}: \mathrm{Set} \longrightarrow \mathrm{Set}_\Delta$ with $X_n = X$, $d_i = \mathrm{id}$, $s_i = \mathrm{id}$.

Proof
$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Set} \arrow[l, shift left, "R"] \end{tikzcd}$$

Let $C^\bullet$ denote the cosimplicial object. Then, for $e \in \mathrm{Set}$,

$$Re_n = \mathrm{Hom}_{Set}(C^{\cdot}([n]), e)$$

and $Re_n = \mathrm{Hom}_{Set_{\Delta 1}}(\Delta 1^n, Re)$

For $X \in Set_{\Delta 1}$,

$$\mathcal{L}X = \int^n X_n \cdot C^{\cdot}([n])$$

where $X_n \cdot C^{\cdot}([n])$

$$= X \cdot C^{\cdot}([n]) = \bigsqcup_x C^{\cdot}([n]).$$

<u>Ex</u> Show that Set is equivalent to $(Set_{\Delta 1})_0$, the category of simplicial sets of dimension 0.

Proof Let X be a simplicial set

$sk^n X = X \Rightarrow \dim(X) = n$

$sk^0 X = X \Rightarrow X$ is just a collection of 0-simplices, and this can trivially be regarded as a set.

Ex To simplicial set X, associate abelian group the $\mathbb{Z}X$ with $\mathbb{Z}X_n$ being the free abelian group on X_n.

Here, $\partial = \sum_{i=0}^{n} (-1)^i d_i$, and the associated chain is called the Moore complex. Show that $\partial^2 = 0$.

<u>Proof</u> $\partial_i = \sum_{i=0}^{n} (-1)^i d_i$

$$\partial_{i-1} = \sum_{i=0}^{n} (-1)^{i-1} d_{i-1}$$

$$\partial_{i-1} \partial_i = \left(\sum_{i=0}^{n} (-1)^{i-1} d_{i-1} \right) \times$$

$$\left(\sum_{i=0}^{n} (-1)^i d_i \right) = 0$$ using the identity $d_i d_j = d_{j-1} d_i$

Ex Show that there exists a fiber bundle of the form $S^1 \longrightarrow S^{2n+1} \longrightarrow\!\!\!\!\!\to \mathbb{P}^n\mathbb{C}$

Proof $\mathbb{P}^n\mathbb{C}$ is obtained from $S^{2n+1} \subset \mathbb{C}^{n+1}$ under the quotient $v \sim \lambda v$, $|\lambda| = 1$
$\mathbb{P}^n\mathbb{C}$ is the base space and S^{2n+1} is the total space of the covering. Let its fiber be F. Then, for open neighbourhood V of $\mathbb{P}^n\mathbb{C}$, given $p : S^{2n+1} \longrightarrow\!\!\!\!\!\to \mathbb{P}^n\mathbb{C}$

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, bend left, "p"'] & & U \times F \arrow[dl, "\mathrm{proj}"] \\
& U &
\end{tikzcd}$$

Indeed, as $\mathbb{P}^n\mathbb{C} = \dfrac{S^{2n+1}}{\substack{v \sim \lambda v, \\ |\lambda| = 1}}$

we have

$$S^1 \longrightarrow S^{2n} \wedge S^1 \twoheadrightarrow \frac{S^{2n+1}}{\substack{v \sim \lambda v \\ |\lambda| = 1}}$$

being the required fiber bundle.

Ex What can be said about $\pi_n(\mathbb{P}^n\mathbb{C})$?

Proof From the complex Hopf fibration, we get the following l.e.s of homotopy groups —

$$(n \geqslant 1) \begin{cases} \pi_1(\mathbb{P}^n\mathbb{C}) = 0, \pi_2 = \mathbb{Z} \\ \pi_{\leq 2k}(\mathbb{P}^n\mathbb{C}) = 0 \\ \pi_{>2k}(\mathbb{P}^n\mathbb{C}) = \pi_n(S^{2n+1}) \end{cases}$$

$$\cdots \longrightarrow \cdots$$

$$0 \to \pi_{2d+2}(S^{2n+1}) \xrightarrow{\cong} \pi_{2d+2}(\mathbb{P}^n\mathbb{C}) \longrightarrow \cdots$$

$$\to 0 \longrightarrow \mathbb{Z} \xrightarrow{\cong} \pi_{2n+1}(\mathbb{P}^n\mathbb{C}) \longrightarrow \cdots$$

$$\to \mathbb{Z} \longrightarrow 0 \longrightarrow \pi_1(\mathbb{P}^n\mathbb{C}) \longrightarrow 0$$

For $n=1$, we have the classic Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow\!\!\!\!\rightarrow S^2 \cong \mathbb{P}^1\mathbb{C}$$

$\pi_0(S^2) = 0$

$\pi_1(S^2) = 0$

$\pi_2(S^2) = \mathbb{Z}$

$\pi_n(\mathbb{P}^1\mathbb{C}) = \pi_n(S^2)$.

<u>Ex</u> Compute $\pi_2(S^2)$ and show that $\pi_n(S^3) \cong \pi_n(S^2)$, $n \geq 3$.

Proof From the classic Hopf fibration,

$$S^1 \longrightarrow S^3 \twoheadrightarrow S^2$$

we have the following l.e.s of homotopy groups –

$\pi_2(S^2) = \mathbb{Z}$

$\pi_n(S^2) \cong \pi_n(S^3)$, $n \geq 3$.

$$\cdots \longrightarrow 0 \longrightarrow \pi_n(S^3) \longrightarrow \pi_n(S^2) \longrightarrow \cdots$$

$$\cdots \longrightarrow 0 \longrightarrow \pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2) \rightarrow \cdots$$

$$\rightarrow 0 \longrightarrow 0 \longrightarrow \pi_2(S^2) \xrightarrow{\cong} \mathbb{Z} \longrightarrow 0$$

Ex What can be said about $\pi_n(\mathbb{P}^1\mathbb{R})$?

Proof $\mathbb{P}^1\mathbb{R} \cong S^1$, and $\pi_n(\mathbb{P}^1\mathbb{R}) = \pi_n(S^1)$.

Ex Compute $\pi_d(\mathbb{R}^n\mathbb{P})$ $\forall d \geqslant 2$ for $n \leqslant d$.

From the real Hopf fibration, $\mathbb{Z}_2 \to S^d \twoheadrightarrow \mathbb{R}^n\mathbb{P}$

$$\left.\begin{aligned} \pi_1(\mathbb{R}^n\mathbb{P}) &= \mathbb{Z}_2 \\ \pi_d(\mathbb{R}^n\mathbb{P}) &= 0 \end{aligned}\right\}.$$

$$\dots \to 0 \to \pi_d(\mathbb{R}^n\mathbb{P}) \to \dots \to 0$$

$$\pi_2(\mathbb{R}^n\mathbb{P}) \to 0 \to \pi_1(\mathbb{R}^n\mathbb{P}) \xrightarrow{\cong} \mathbb{Z}_2 \to 0$$

Ex What can be said

about $\pi_d(\mathbb{R}^n\mathbb{P})$, $\forall d \geq 2$ and $n > d$?

Proof $\pi_d(\mathbb{R}^n\mathbb{P}) \cong \pi_d(S^n)$ from the real Hopf fibration.

Ex Show that S^∞ is contractible.

Proof A space is contractible iff $\pi_n(X) = 0$ $\forall n$. S^n has a CW-decomposition as two j-cells, $\forall j \leq n$. $S^0 \subset S^1 \subset \ldots \subset S^\infty$.

$$\pi_k(S^n) = 0 \quad \forall\, k < n$$

$$\operatorname*{colim}_{n \to \infty} \pi_k(S^n) = \pi_k(S^\infty)$$
$$= 0 \quad \forall\, k.$$

Ex Is $\mathbb{R}^n$ a CW-complex? Is it finite?

Proof $\mathbb{R}^n$ has the cellular decomposition $\mathbb{Z}^n$ 0-cells and $\mathbb{Z}^{n+2}$ 1-cells, with the two extra copies of $\mathbb{Z}$ acting as the boundary of the attaching map. It is not finite because $\bigsqcup_{n \in \mathbb{N}} J_n$ is $\mathbb{Z}^n$ or $\mathbb{Z}^{n+2}$, which are not finite.

Ex Prove that $\pi_n(S^n) \cong \mathbb{Z}$

Proof By the Freudenthal suspension theorem, in any non-degenerately based $(n-1)$-connected space, $n \geq 1$, the Σ functor

$$\Sigma : \pi_q(X) \longrightarrow \pi_{q+1}(\Sigma X)$$

is a bijection for $q < 2n+1$

and surjection for $q = 2n-1$.

Hence, for $n > 1$

$$\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$$

From the complex Hopf bundle

$$S^1 \longrightarrow S^3 \twoheadrightarrow S^2,$$

$$\begin{tikzcd}
\pi_2(S^2) \arrow[r, "\cong"] & \pi_1(S^1) \arrow[r] & 0 \\
0 \arrow[u] & &
\end{tikzcd}$$

Since $\pi_1(S^1) = \mathbb{Z} \cong \pi_2(S^2)$, we have proved that

$\pi_n(S^n) \cong \mathbb{Z}$.

Ex Show that S^n admits a CW-complex structure with 2 cells in every dimension $k \leq n$.

Proof

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & \emptyset \arrow[d, tail] \\
\{*\} \sqcup \{*\} \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & X^{(0)}
\end{tikzcd}$$

$X^{(0)} = \{*\}$

$$\begin{tikzcd}
S^0 \sqcup S^0 \arrow[r, "f"] \arrow[d, tail] & \{*\} \arrow[d, tail] \\
D^1 \sqcup D^1 \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & X^{(1)}
\end{tikzcd}$$

$X^{(1)} = D^1 \sqcup_f \{*\}$

$$\begin{tikzcd}
S^1 \sqcup S^1 \arrow[r, "g"] \arrow[d, tail] & P^1 \sqcup \{*\} \arrow[d, tail] \\
P^2 \sqcup P^2 \arrow[r] & X^{(2)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$$X^{(2)} = (P^2 \sqcup P^2) \sqcup_g (P^1 \sqcup \{*\})$$
$$= P^2 \sqcup_g \{*\}$$

$$X^{(n)} = P^n \sqcup_h \{*\}$$

<u>Ex</u> Let X be the colim $X = \operatorname{colim}_{n \in \mathbb{N}} X^{(n)}$, and let each $X^{(n)} \rightarrowtail X^{(n+1)}$ be nullhomotopic. Show that

X is contractible.

Proof $X^{(n+1)} = \coprod_{J_{n+1}} D^{n+1} \sqcup_f X^{(n)}$

$$\begin{tikzcd}
\coprod_{J_n} S^{n-1} \arrow[r, "f"'] \arrow[dr, "\mathrm{IS}", "f"'] \arrow[ddr, "\mathrm{IS}", "f"'] & X^{(n)} \arrow[d, "0"] \\
 & X^{(n+1)} \arrow[d, "0"] \\
 & X^{(n+2)}
\end{tikzcd}$$

where $X^{(0)} = \coprod_{J_0} \{*\}$

Consider $X^{(0)} \xrightarrow{0} X^{(n+1)}$

$$\coprod_{J_0} D^0 \xrightarrow{0} \coprod_{J_{n+1}} D^{n+1} \sqcup_f X^{(n)}$$

$\Rightarrow J_n = J_{n+1} \; \forall n.$

Consider the case:

$$\begin{tikzcd}
\bigsqcup_J S^0 \arrow[r] \arrow[dr, "1S"] & X^{(1)} \arrow[d, "0"] \\
& X^{(3)}
\end{tikzcd}$$

A non-trivial space can be constructed with $J=2$ and at a minimum, this space is S^∞.
The space is obviously contractible, as there are J cells in every dimension, with the coproduct of n-dimensional disks

attached to the boundary of the coproduct of $(n-1)$-dimensional disks. When the process is stopped at a finite n, we have nonzero π_k's for $k \geq n$, but when passing to the colimit, we get $\pi_k = 0 \ \forall k$, making the space contractible.