$H_n = \text{Ker}\,\partial_n / \text{Im}\,\partial_{n+1}$ boundaries

cycles

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

The algebraic situation we have now is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$\text{Ker}\,\partial_0 = \mathbb{Z}$

each

with $\partial_n \partial_{n+1} = 0$ for each n. Such a sequence is called a **chain complex**. Note that we have extended the sequence by a 0 at the right end, with $\partial_0 = 0$. From $\partial_n\partial_{n+1} = 0$ it follows that $\text{Im}\,\partial_{n+1} \subset \text{Ker}\,\partial_n$, where Im and Ker denote image and kernel. So we can define the n^{th} **homology group** of the chain complex to be the quotient group $H_n = \text{Ker}\,\partial_n / \text{Im}\,\partial_{n+1}$. Elements of $\text{Ker}\,\partial_n$ are called **cycles** and elements of $\text{Im}\,\partial_{n+1}$ are **boundaries**. Elements of H_n are cosets of $\text{Im}\,\partial_{n+1}$, called **homology classes**. Two cycles representing the same homology class are said to be **homologous**. This means their difference is a boundary.

$\text{im}\,\partial_2 = 0$

$\text{Ker}\,\partial_2 = \mathbb{Z}$

$C_n = \Delta_n(X)$

$= \sum n_\alpha \sigma_\alpha$

Returning to the case that $C_n = \Delta_n(X)$, the homology group $\text{Ker}\,\partial_n / \text{Im}\,\partial_{n+1}$ will be denoted $H_n^\Delta(X)$ and called the n^{th} **simplicial homology group** of X.

$\text{Ker}\,\partial_1 = \mathbb{Z} \oplus \mathbb{Z}$

$\sigma_\alpha : \Delta^n \to X$

Example 2.2. $X = S^1$, with one vertex v and one edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both $\mathbb{Z}$ and the boundary map im ∂_1 is zero since $\partial e = v - v$. The groups $\Delta_n(S^1)$ are 0 for $n \ge 2$ since there are no simplices in these dimensions. Hence

$$H_n^\Delta(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \ge 2 \end{cases}$$

$\text{im}\,\partial_1 = 0$

$\partial_1 :$

$\partial_1 :$

$\Delta_1 \to \Delta_0$

This is an illustration of the general fact that if the boundary maps in a chain complex are all zero, then the homology groups of the complex are isomorphic to the chain groups themselves.

$[c]$

$\partial_1 = 0$

Example 2.3. $X = T$, the torus with the Δ-complex structure pictured earlier, having one vertex, three edges a, b, and c, and two 2-simplices U and L. ~~As in the previous example~~, im $\partial_1 = 0$ so $H_0^\Delta(T) \approx \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a+b-c\}$ is a basis for $\Delta_1(T)$, it follows that $H_1^\Delta(T) \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^\Delta(T)$ is equal to $\text{Ker}\,\partial_2$, which is infinite cyclic generated by $U - L$ since $\partial(pU + qL) = (p+q)(a+b-c) = 0$ only if $p = -q$. Thus

$[a+b]$

$\partial_2 :$

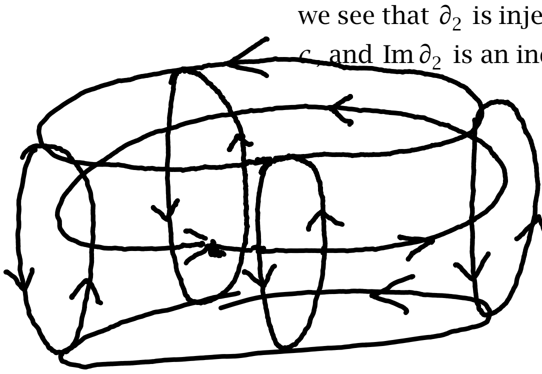
$\Delta_2 \to \Delta_1$

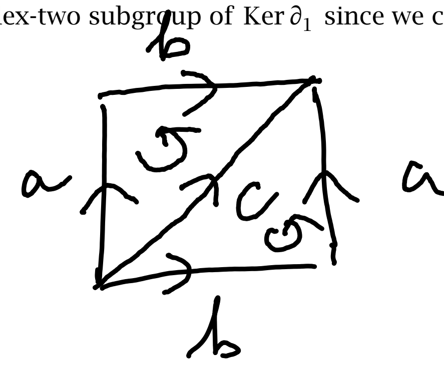
$$H_n^\Delta(T) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \ge 3 \end{cases}$$

$[v_+]\ [v_-]$

Example 2.4. $X = \mathbb{R}\mathrm{P}^2$, as pictured earlier, with two vertices v and w, three edges a, b, and c, and two 2-simplices U and L. Then $\text{Im}\,\partial_1$ is generated by $w - v$, so $H_0^\Delta(X) \approx \mathbb{Z}$ with either vertex as a generator. Since $\partial_2 U = -a+b+c$ and $\partial_2 L = a-b+c$, we see that ∂_2 is injective, so $H_2^\Delta(X) = 0$. Further, $\text{Ker}\,\partial_1 \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis $a - b$ and c, and $\text{Im}\,\partial_2$ is an index-two subgroup of $\text{Ker}\,\partial_1$ since we can choose c and $a - b + c$

Δ_1





$\partial_2 : [-b - a + c]$

$[-c + \cancel{b+a}]$

as a basis for $\operatorname{Ker}\partial_1$ and $a-b+c$ and $2c=(a-b+c)+(-a+b+c)$ as a basis for $\operatorname{Im}\partial_2$. Thus $H_1^\Delta(X)\approx\mathbb{Z}_2$.

Example 2.5. We can obtain a Δ-complex structure on S^n by taking two copies of Δ^n and identifying their boundaries via the identity map. Labeling these two n-simplices U and L, then it is obvious that $\operatorname{Ker}\partial_n$ is infinite cyclic generated by $U-L$. Thus $H_n^\Delta(S^n)\approx\mathbb{Z}$ for this Δ-complex structure on S^n. Computing the other homology groups would be more difficult.

Many similar examples could be worked out without much trouble, such as the other closed orientable and nonorientable surfaces. However, the calculations do tend to increase in complexity before long, particularly for higher-dimensional complexes.

Some obvious general questions arise: Are the groups $H_n^\Delta(X)$ independent of the choice of Δ-complex structure on X? In other words, if two Δ-complexes are homeomorphic, do they have isomorphic homology groups? More generally, do they have isomorphic homology groups if they are merely homotopy equivalent? To answer such questions and to develop a general theory it is best to leave the rather rigid simplicial realm and introduce the singular homology groups. These have the added advantage that they are defined for all spaces, not just Δ-complexes. At the end of this section, after some theory has been developed, we will show that simplicial and singular homology groups coincide for Δ-complexes.

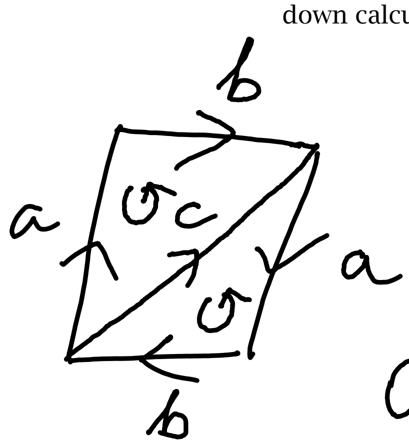
Traditionally, simplicial homology is defined for **simplicial complexes**, which are the Δ-complexes whose simplices are uniquely determined by their vertices. This amounts to saying that each n-simplex has $n+1$ distinct vertices, and that no other n-simplex has this same set of vertices. Thus a simplicial complex can be described combinatorially as a set X_0 of vertices together with sets X_n of n-simplices, which are $(n+1)$-element subsets of X_0. The only requirement is that each $(k+1)$-element subset of the vertices of an n-simplex in X_n is a k-simplex, in X_k. From this combinatorial data a Δ-complex X can be constructed, once we choose a partial ordering of the vertices X_0 that restricts to a linear ordering on the vertices of each simplex in X_n. For example, we could just choose a linear ordering of all the vertices. This might perhaps involve invoking the Axiom of Choice for large vertex sets.

Ker ∂_1 $=\mathbb{Z}\oplus\mathbb{Z}$

An exercise at the end of this section is to show that every Δ-complex can be subdivided to be a simplicial complex. In particular, every Δ-complex is then homeomorphic to a simplicial complex.

Ker ∂_2 $=0$

Compared with simplicial complexes, Δ-complexes have the advantage of simpler computations since fewer simplices are required. For example, to put a simplicial complex structure on the torus one needs at least 14 triangles, 21 edges, and 7 vertices, and for $\mathbb{R}P^2$ one needs at least 10 triangles, 15 edges, and 6 vertices. This would slow down calculations considerably!



$\mathbb{R}P^2$

2×0s x

3×1s x

2×2s x

$[b-a]$

$[c]$

$[c-b-a]$

$[a+b+c]$

$0\xrightarrow{\partial_2}\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{\partial_1}\mathbb{Z}$

$\deg f = \deg g \Longleftrightarrow f \simeq g$

2.2 Computations and Applications

Now that the basic properties of homology have been established, we can begin to move a little more freely. Our first topic, exploiting the calculation of $H_n(S^n)$, is Brouwer's notion of degree for maps $S^n \to S^n$. Historically, Brouwer's introduction of this concept in the years 1910–12 preceded the rigorous development of homology, so his definition was rather different, using the technique of simplicial approximation which we explain in §2.C. The later definition in terms of homology is certainly more elegant, though perhaps with some loss of geometric intuition. More in the spirit of Brouwer's definition is a third approach using differential topology, presented very lucidly in [Milnor 1965].

Degree

For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on f. This integer is called the **degree** of f, with the notation $\deg f$. Here are some basic properties of degree:

(a) $\deg \mathbb{1} = 1$, since $\mathbb{1}_* = \mathbb{1}$.

(b) $\deg f = 0$ if f is not surjective. For if we choose a point $x_0 \in S^n - f(S^n)$ then f can be factored as a composition $S^n \to S^n - \{x_0\} \hookrightarrow S^n$ and $H_n(S^n - \{x_0\}) = 0$ since $S^n - \{x_0\}$ is contractible. Hence $f_* = 0$.

$S^n - \{x_0\}$ is homeo. to a disc, which is contractible

(c) If $f \simeq g$ then $\deg f = \deg g$ since $f_* = g_*$. The converse statement, that $f \simeq g$ if $\deg f = \deg g$, is a fundamental theorem of Hopf from around 1925 which we prove in Corollary 4.25.

(d) $\deg fg = \deg f \deg g$, since $(fg)_* = f_* g_*$. As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence since $fg \simeq \mathbb{1}$ implies $\deg f \deg g = \deg \mathbb{1} = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n, fixing the points in a subsphere S^{n-1} and interchanging the two complementary hemispheres. For we can give S^n a Δ-complex structure with these two hemispheres as its two n-simplices Δ_1^n and Δ_2^n, and the n-chain $\Delta_1^n - \Delta_2^n$ represents a generator of $H_n(S^n)$ as we saw in Example 2.23, so the reflection interchanging Δ_1^n and Δ_2^n sends this generator to its negative.

(f) The antipodal map $-\mathbb{1} : S^n \to S^n$, $x \mapsto -x$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$. $n+1$ mirrors on axes of refl.

(g) If $f : S^n \to S^n$ has no fixed points then $\deg f = (-1)^{n+1}$. For if $f(x) \neq x$ then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \le t \le 1$, does not pass through the origin. Hence if f has no fixed points, the formula $f_t(x) = [(1-t)f(x) - tx]/|(1-t)f(x) - tx|$ defines a homotopy from f to

$\deg(fg) = (\deg f)(\deg g)$

$\deg : S^n \xrightarrow{d} S^n$. $d \in \mathbb{Z}$

the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal map is a sort of converse statement.

Here is an interesting application of degree:

Theorem 2.28. *S^n has a continuous field of nonzero tangent vectors iff n is odd.*

Proof: Suppose $x \mapsto v(x)$ is a tangent vector field on S^n, assigning to a vector $x \in S^n$ the vector $v(x)$ tangent to S^n at x. Regarding $v(x)$ as a vector at the origin instead of at x, tangency just means that x and $v(x)$ are orthogonal in $\mathbb{R}^{n+1}$. If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Assuming this has been done, the vectors $(\cos t)x + (\sin t)v(x)$ lie in the unit circle in the plane spanned by x and $v(x)$. Letting t go from 0 to π, we obtain a homotopy $f_t(x) = (\cos t)x + (\sin t)v(x)$ from the identity map of S^n to the antipodal map $-\mathbb{1}$. This implies that $\deg(-\mathbb{1}) = \deg \mathbb{1}$, hence $(-1)^{n+1} = 1$ and n must be odd.

Conversely, if n is odd, say $n = 2k-1$, we can define $v(x_1, x_2, \cdots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$. Then $v(x)$ is orthogonal to x, so v is a tangent vector field on S^n, and $|v(x)| = 1$ for all $x \in S^n$. □

For the much more difficult problem of finding the maximum number of tangent vector fields on S^n that are linearly independent at each point, see [VBKT] or [Husemoller 1966].

Another nice application of degree, giving a partial answer to a question raised in Example 1.43, is the following result:

Proposition 2.29. *$\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.*

Recall that an action of a group G on a space X is a homomorphism from G to the group $\mathrm{Homeo}(X)$ of homeomorphisms $X \to X$, and the action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points. In the case of S^n, the antipodal map $x \mapsto -x$ generates a free action of $\mathbb{Z}_2$.

Proof: Since the degree of a homeomorphism must be ± 1, an action of a group G on S^n determines a degree function $d: G \to \{\pm 1\}$. This is a homomorphism since $\deg fg = \deg f \deg g$. If the action is free, then d sends every nontrivial element of G to $(-1)^{n+1}$ by property (g) above. Thus when n is even, d has trivial kernel, so $G \subset \mathbb{Z}_2$. □

There are two maps $\{0\} \to \{0\}$ and $(G - \{0\}) \to \{-1\}$

Next we describe a technique for computing degrees which can be applied to most maps that arise in practice. Suppose $f: S^n \to S^n$, $n > 0$, has the property that for

deg of a homeo is ± 1, since $d: S^n \to S^n$ must be a homotopy equiv.

some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say $x_1, \cdots, x_m$. Let $U_1, \cdots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y. Then $f(U_i - x_i) \subset V - y$ for each i, and we have a commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\approx"'] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\approx"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"] \arrow[ul, "\approx"] & H_n(S^n) \arrow[u, "\approx"]
\end{tikzcd}$$

where all the maps are the obvious ones, in particular k_i and p_i are induced by inclusions. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Via these four isomorphisms, the top two groups in the diagram can be identified with $H_n(S^n) \approx \mathbb{Z}$, and the top homomorphism f_* becomes multiplication by an integer called the **local degree** of f at x_i, written $\deg f \,|\, x_i$.

For example, if f is a homeomorphism, then y can be any point and there is only one corresponding x_i, so all the maps in the diagram are isomorphisms and $\deg f \,|\, x_i = \deg f = \pm 1$. More generally, if f maps each U_i homeomorphically onto V, then $\deg f \,|\, x_i = \pm 1$ for each i. This situation occurs quite often in applications, and it is usually not hard to determine the correct signs.

Here is the formula that reduces degree calculations to computing local degrees:

Proposition 2.30. $\deg f = \sum_i \deg f \,|\, x_i$.

Proof: By excision, the central term $H_n(S^n, S^n - f^{-1}(y))$ in the preceding diagram is the direct sum of the groups $H_n(U_i, U_i - x_i) \approx \mathbb{Z}$, with k_i the inclusion of the i^{th} summand. Since the upper triangle commutes, the projections of this direct sum onto its summands are given by the maps p_i. Identifying the outer groups in the diagram with $\mathbb{Z}$ as before, commutativity of the lower triangle says that $p_i j(1) = 1$, hence $j(1) = (1, \cdots, 1) = \sum_i k_i(1)$. Commutativity of the upper square says that the middle f_* takes $k_i(1)$ to $\deg f \,|\, x_i$, hence $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f \,|\, x_i$. Commutativity of the lower square then gives the formula $\deg f = \sum_i \deg f \,|\, x_i$. $\square$

Example 2.31. We can use this result to construct a map $S^n \to S^n$ of any given degree, for each $n \geq 1$. Let $q: S^n \to \bigvee_k S^n$ be the quotient map obtained by collapsing the complement of k disjoint open balls B_i in S^n to a point, and let $p: \bigvee_k S^n \to S^n$ identify all the summands to a single sphere. Consider the composition $f = pq$. For almost all $y \in S^n$ we have $f^{-1}(y)$ consisting of one point x_i in each B_i. The local degree of f at x_i is ± 1 since f is a homeomorphism near x_i. By precomposing p with reflections of the summands of $\bigvee_k S^n$ if necessary, we can make each local degree either $+1$ or -1, whichever we wish. Thus we can produce a map $S^n \to S^n$ of degree $\pm k$.

$\vee_k S^n \to S^n$

has degree $\pm k$

$\deg(\Sigma f) = \deg f$

Example 2.32. In the case of S^1, the map $f(z) = z^k$, where we view S^1 as the unit circle in $\mathbb{C}$, has degree k. This is evident in the case $k = 0$ since f is then constant. The case $k < 0$ reduces to the case $k > 0$ by composing with $z \mapsto z^{-1}$, which is a reflection, of degree -1. To compute the degree when $k > 0$, observe first that for any $y \in S^1$, $f^{-1}(y)$ consists of k points $x_1, \cdots, x_k$ near each of which f is a local homeomorphism, stretching a circular arc by a factor of k. This local stretching can be eliminated by a deformation of f near x_i that does not change local degree, so the local degree at x_i is the same as for a rotation of S^1. A rotation is a homeomorphism so its local degree at any point equals its global degree, which is $+1$ since a rotation is homotopic to the identity. Hence $\deg f\,|\,x_i = 1$ and $\deg f = k$.

Another way of obtaining a map $S^n \to S^n$ of degree k is to take a repeated suspension of the map $z \mapsto z^k$ in Example 2.32, since suspension preserves degree:

Proposition 2.33. *$\deg Sf = \deg f$, where $Sf : S^{n+1} \to S^{n+1}$ is the suspension of the map $f : S^n \to S^n$.*

Proof: Let CS^n denote the cone $(S^n \times I)/(S^n \times 1)$ with base $S^n = S^n \times 0 \subset CS^n$, so CS^n/S^n is the suspension of S^n. The map f induces $Cf : (CS^n, S^n) \to (CS^n, S^n)$ with quotient Sf. The naturality of the boundary maps in the long exact sequence of the pair (CS^n, S^n) then gives commutativity of the diagram at the right. Hence if f_* is multiplication by d, so is Sf_*. $\square$

$$\begin{tikzcd} \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\approx"'] \arrow[d, "Sf_*"] & \widetilde{H}_n(S^n) \arrow[d, "f_*"] \\ \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\approx"'] & \widetilde{H}_n(S^n) \end{tikzcd}$$

Note that for $f : S^n \to S^n$, the suspension Sf maps only one point to each of the two 'poles' of S^{n+1}. This implies that the local degree of Sf at each pole must equal the global degree of Sf. Thus the local degree of a map $S^n \to S^n$ can be any integer if $n \ge 2$, just as the degree itself can be any integer when $n \ge 1$.

Cellular Homology

Cellular homology is a very efficient tool for computing the homology groups of CW complexes, based on degree calculations. Before giving the definition of cellular homology, we first establish a few preliminary facts:

Lemma 2.34. *If X is a CW complex, then:*

(a) *$H_k(X^n, X^{n-1})$ is zero for $k \ne n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n-cells of X.*

(b) *$H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.*

(c) *The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.*

Proof: Statement (a) follows immediately from the observation that (X^n, X^{n-1}) is a good pair and X^n/X^{n-1} is a wedge sum of n-spheres, one for each n-cell of X. Here we are using Proposition 2.22 and Corollary 2.25.

$$\frac{X^n}{X^{n-1}} = \bigvee_k S^n$$

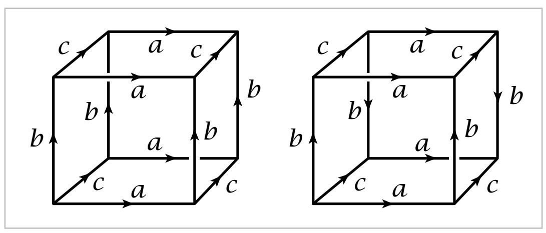
These two examples illustrate the general fact that the orientability of a closed connected manifold M of dimension n is detected by $H_n(M)$, which is $\mathbb{Z}$ if M is orientable and 0 otherwise. This is shown in Theorem 3.26.

Example 2.38: An Acyclic Space. Let X be obtained from $S^1 \vee S^1$ by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$. Then $d_2 : \mathbb{Z}^2 \to \mathbb{Z}^2$ has matrix $\left(\begin{smallmatrix} 5 & -2 \\ -3 & 1 \end{smallmatrix}\right)$, with the two columns coming from abelianizing a^5b^{-3} and $b^3(ab)^{-2}$ to $5a - 3b$ and $-2a + b$, in additive notation. The matrix has determinant -1, so d_2 is an isomorphism and $\widetilde{H}_i(X) = 0$ for all i. Such a space X is called **acyclic**.

We can see that this acyclic space is not contractible by considering $\pi_1(X)$, which has the presentation $\langle a, b \mid a^5b^{-3}, b^3(ab)^{-2} \rangle$. There is a nontrivial homomorphism from this group to the group G of rotational symmetries of a regular dodecahedron, sending a to the rotation ρ_a through angle $2\pi/5$ about the axis through the center of a pentagonal face, and b to the rotation ρ_b through angle $2\pi/3$ about the axis through a vertex of this face. The composition $\rho_a\rho_b$ is a rotation through angle π about the axis through the midpoint of an edge abutting this vertex. Thus the relations $a^5 = b^3 = (ab)^2$ defining $\pi_1(X)$ become $\rho_a^5 = \rho_b^3 = (\rho_a\rho_b)^2 = 1$ in G, which means there is a well-defined homomorphism $\rho : \pi_1(X) \to G$ sending a to ρ_a and b to ρ_b. It is not hard to see that G is generated by ρ_a and ρ_b, so ρ is surjective. With more work one can compute that the kernel of ρ is $\mathbb{Z}_2$, generated by the element $a^5 = b^3 = (ab)^2$, and this $\mathbb{Z}_2$ is in fact the center of $\pi_1(X)$. In particular, $\pi_1(X)$ has order 120 since G has order 60.

After these 2-dimensional examples, let us now move up to three dimensions, where we have the additional task of computing the cellular boundary map d_3.

Example 2.39. A 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ can be constructed from a cube by identifying each pair of opposite square faces as in the first of the two figures. The second figure shows a slightly different pattern of identifications of opposite faces, with the front and back faces now identified via a rotation of the cube around a horizontal left-right axis. The space produced by these identifications is the product $K \times S^1$ of a Klein bottle and a circle. For both T^3 and $K \times S^1$ we have a CW structure with one 3-cell, three 2-cells, three 1-cells, and one 0-cell. The cellular chain complexes thus have the form



$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

In the case of the 3-torus T^3 the cellular boundary map d_2 is zero by the same calculation as for the 2-dimensional torus. We claim that d_3 is zero as well. This amounts to saying that the three maps $\Delta_{\alpha\beta} : S^2 \to S^2$ corresponding to the three 2-cells local deg. or ✓

have degree zero. Each $\Delta_{\alpha\beta}$ maps the interiors of two opposite faces of the cube homeomorphically onto the complement of a point in the target S^2 and sends the remaining four faces to this point. Computing local degrees at the center points of the two opposite faces, we see that the local degree is $+1$ at one of these points and -1 at the other, since the restrictions of $\Delta_{\alpha\beta}$ to these two faces differ by a reflection of the boundary of the cube across the plane midway between them, and a reflection has degree -1. Since the cellular boundary maps are all zero, we deduce that $H_i(T^3)$ is $\mathbb{Z}$ for $i = 0, 3$, $\mathbb{Z}^3$ for $i = 1, 2$, and 0 for $i > 3$.

* For $K \times S^1$, when we compute local degrees for the front and back faces we find that the degrees now have the same rather than opposite signs since the map $\Delta_{\alpha\beta}$ on these two faces differs not by a reflection but by a rotation of the boundary of the cube. The local degrees for the other faces are the same as before. Using the letters A, B, C to denote the 2-cells given by the faces orthogonal to the edges a, b, c, respectively, we have the boundary formulas $d_3 e^3 = 2C$, $d_2 A = 2b$, $d_2 B = 0$, and $d_2 C = 0$. It follows that $H_3(K \times S^1) = 0$, $H_2(K \times S^1) = \mathbb{Z} \oplus \mathbb{Z}_2$, and $H_1(K \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

$\dfrac{\operatorname{Ker} d_2}{\operatorname{im} d_3} = \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z}_2$

$d_1 = 0$

Many more examples of a similar nature, quotients of a cube or other polyhedron with faces identified in some pattern, could be worked out in similar fashion. But let us instead turn to some higher-dimensional examples.

Example 2.40: Moore Spaces. Given an abelian group G and an integer $n \ge 1$, we will construct a CW complex X such that $H_n(X) \approx G$ and $\widetilde{H}_i(X) = 0$ for $i \ne n$. Such a space is called a **Moore space**, commonly written $M(G, n)$ to indicate the dependence on G and n. It is probably best for the definition of a Moore space to include the condition that $M(G, n)$ be simply-connected if $n > 1$. The spaces we construct will have this property.

As an easy special case, when $G = \mathbb{Z}_m$ we can take X to be S^n with a cell e^{n+1} attached by a map $S^n \to S^n$ of degree m. More generally, any finitely generated G can be realized by taking wedge sums of examples of this type for finite cyclic summands of G, together with copies of S^n for infinite cyclic summands of G.

In the general nonfinitely generated case let $F \to G$ be a homomorphism of a free abelian group F onto G, sending a basis for F onto some set of generators of G. The kernel K of this homomorphism is a subgroup of a free abelian group, hence is itself free abelian. Choose bases $\{x_\alpha\}$ for F and $\{y_\beta\}$ for K, and write $y_\beta = \sum_\alpha d_{\beta\alpha} x_\alpha$. Let $X^n = \bigvee_\alpha S^n_\alpha$, so $H_n(X^n) \approx F$ via Corollary 2.25. We will construct X from X^n by attaching cells e^{n+1}_β via maps $f_\beta : S^n \to X^n$ such that the composition of f_β with the projection onto the summand S^n_α has degree $d_{\beta\alpha}$. Then the cellular boundary map d_{n+1} will be the inclusion $K \hookrightarrow F$, hence X will have the desired homology groups.

$\dfrac{\operatorname{Ker} d_1}{\operatorname{im} d_2} = \dfrac{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}{2\mathbb{Z}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$

The construction of f_β generalizes the construction in Example 2.31 of a map $S^n \to S^n$ of given degree. Namely, we can let f_β map the complement of $\sum_\alpha |d_{\beta\alpha}|$

$\dfrac{\mathbb{Z}_4 \oplus \mathbb{Z}_4}{\mathbb{Z}_2} = \mathbb{Z}_4 \oplus \mathbb{Z}_2$

3.4. Carry out a similar exercise to the one above, assuming α'' is an isomorphism.
3.5. Use the *universal property* of the direct sum to show that

$$(A_1 \oplus A_2) \oplus A_3 \cong A_1 \oplus (A_2 \oplus A_3).$$

3.6. Show that $\mathbb{Z}_m \oplus \mathbb{Z}_n = \mathbb{Z}_{mn}$ if and only if m and n are mutually prime.
3.7. Show that the following statements about the exact sequence

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$$

of Λ-modules are equivalent:
(i) there exists $\mu: A'' \to A$ with $\alpha''\mu = 1$ on A'';
(ii) there exists $\varepsilon: A \to A'$ with $\varepsilon\alpha' = 1$ on A';
(iii) $0 \to \mathrm{Hom}_\Lambda(B, A') \xrightarrow{\alpha'_*} \mathrm{Hom}_\Lambda(B, A) \xrightarrow{\alpha''_*} \mathrm{Hom}_\Lambda(B, A'') \to 0$ is exact for all B;
(iv) $0 \to \mathrm{Hom}_\Lambda(A'', C) \xrightarrow{\alpha''^*} \mathrm{Hom}_\Lambda(A, C) \xrightarrow{\alpha'^*} \mathrm{Hom}_\Lambda(A', C) \to 0$ is exact for all C;
(v) there exists $\mu: A'' \to A$ such that $\langle \alpha', \mu \rangle: A' \oplus A'' \xrightarrow{\sim} A$.

3.8. Show that if $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ is pure and if A'' is a direct sum of cyclic groups then statement (i) above holds (see Exercise 2.7).

4. Free and Projective Modules

Let A be a Λ-module and let S be a subset of A. We consider the set A_0 of all elements $a \in A$ of the form $a = \sum_{s \in S} \lambda_s s$ where $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is trivially seen that A_0 is a submodule of A; hence it is the smallest submodule of A containing S.

If for the set S the submodule A_0 is the whole of A, we shall say that S is a set of *generators* of A. If A admits a finite set of generators it is said to be *finitely generated*. A set S of generators of A is called a *basis* of A if every element $a \in A$ may be expressed *uniquely* in the form $a = \sum_{s \in S} \lambda_s s$ with $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is readily seen that a set S of generators is a basis if and only if it is *linearly independent*, that is, if $\sum_{s \in S} \lambda_s s = 0$ implies $\lambda_s = 0$ for all $s \in S$. The reader should note that not every module possesses a basis.

Definition. If S is a basis of the Λ-module P, then P is called *free on the set S*. We shall call P *free* if it is free on some subset.

Proposition 4.1. *Suppose the Λ-module P is free on the set S. Then $P \cong \bigoplus_{s \in S} \Lambda_s$ where $\Lambda_s = \Lambda$ as a left module for $s \in S$. Conversely, $\bigoplus_{s \in S} \Lambda_s$ is free on the set $\{1_{\Lambda_s}, s \in S\}$.*

Proof. We define $\varphi: P \to \bigoplus_{s \in S} \Lambda_s$ as follows: Every element $a \in P$ is expressed uniquely in the form $a = \sum_{s \in S} \lambda_s s$; set $\varphi(a) = (\lambda_s)_{s \in S}$. Conversely,

$f : S \longrightarrow$ set underly. MU

free on set $\subset P \longrightarrow M$

for $s \in S$ define $\psi_s : \Lambda_s \to P$ by $\psi_s(\lambda_s) = \lambda_s s$. By the universal property of the direct sum the family $\{\psi_s\}$, $s \in S$, gives rise to a map $\psi = \langle \psi_s \rangle : \bigoplus_{s \in S} \Lambda_s \to P$. It is readily seen that φ and ψ are inverse to each other. The remaining assertion immediately follows from the construction of the direct sum. $\square$

The next proposition yields a universal characterization of the free module on the set S.

Proposition 4.2. *Let P be free on the set S. To every Λ-module M and to every function f from S into the set underlying M, there is a unique Λ-module homomorphism $\varphi : P \to M$ extending f.*

Proof. Let $f(s) = m_s$. Set $\varphi(a) = \varphi\left(\sum_{s \in S} \lambda_s s\right) = \sum_{s \in S} \lambda_s m_s$. This obviously is the only homomorphism having the required property. $\square$

Proposition 4.3. *Every Λ-module A is a quotient of a free module P.*

Proof. Let S be a set of generators of A. Let $P = \bigoplus_{s \in S} \Lambda_s$ with $\Lambda_s = \Lambda$ and define $\varphi : P \to A$ to be the extension of the function f given by $f(1_{\Lambda_s}) = s$. Trivially φ is surjective. $\square$

Proposition 4.4. *Let P be a free Λ-module. To every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$.*

Proof. Let P be free on S. Since ε is surjective we can find elements $b_s \in B$, $s \in S$ with $\varepsilon(b_s) = \gamma(s)$, $s \in S$. Define β as the extension of the function $f : S \to B$ given by $f(s) = b_s$, $s \in S$. By the uniqueness part of Proposition 4.2 we conclude that $\varepsilon\beta = \gamma$. $\square$

To emphasize the importance of the property proved in Proposition 4.4 we make the following remark: Let $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ be a short exact sequence of Λ-modules. If P is a free Λ-module Proposition 4.4 asserts that every homomorphism $\gamma : P \to C$ is induced by a homomorphism $\beta : P \to B$. Hence using Theorem 2.1 we can conclude that the induced sequence

$$0 \to \mathrm{Hom}_\Lambda(P, A) \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(P, C) \to 0 \tag{4.1}$$

is exact, i.e. that ε_* is surjective. Conversely, it is readily seen that exactness of (4.1) for all short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ implies for the module P the property asserted in Proposition 4.4 for P a free module. Therefore there is considerable interest in the class of modules having this property. These are by definition the projective modules:

Definition. A Λ-module P is *projective* if to every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ with $\varepsilon\beta = \gamma$. Equivalently, to any homomorphisms ε, γ with ε surjective in the diagram below there exists every free module is projective

β such that the triangle

$$\begin{tikzcd} & P \arrow[dl, dotted, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, two heads, "\varepsilon"] & C \end{tikzcd}$$

is commutative.

As mentioned above, every free module is projective. We shall give some more examples of projective modules at the end of this section.

Proposition 4.5. *A direct sum $\bigoplus_{i \in I} P_i$ is projective if and only if each P_i is.*

Proof. We prove the proposition only for $A = P \oplus Q$. The proof in the general case is analogous. First assume P and Q projective. Let $\varepsilon : B \twoheadrightarrow C$ be surjective and $\gamma : P \oplus Q \to C$ a homomorphism. Define $\gamma_P = \gamma\,\iota_P : P \to C$ and $\gamma_Q = \gamma\,\iota_Q : Q \to C$. Since P, Q are projective there exist β_P, β_Q such that $\varepsilon\beta_P = \gamma_P$, $\varepsilon\beta_Q = \gamma_Q$. By the universal property of the direct sum there exists $\beta : P \oplus Q \to B$ such that $\beta\,\iota_P = \beta_P$ and $\beta\,\iota_Q = \beta_Q$. It follows that $(\varepsilon\beta)\,\iota_P = \varepsilon\beta_P = \gamma_P = \gamma\,\iota_P$ and $(\varepsilon\beta)\,\iota_Q = \varepsilon\beta_Q = \gamma_Q = \gamma\,\iota_Q$. By the uniqueness part of the universal property we conclude that $\varepsilon\beta = \gamma$. Of course, this could be proved using the explicit *construction* of $P \oplus Q$, but we prefer to emphasize the universal property of the direct sum.

Next assume that $P \oplus Q$ is projective. Let $\varepsilon : B \twoheadrightarrow C$ be a surjection and $\gamma_P : P \to C$ a homomorphism. Choose $\gamma_Q : Q \to C$ to be the zero map. We obtain $\gamma : P \oplus Q \to C$ such that $\gamma\,\iota_P = \gamma_P$ and $\gamma\,\iota_Q = \gamma_Q = 0$. Since $P \oplus Q$ is projective there exists $\beta : P \oplus Q \to B$ such that $\varepsilon\beta = \gamma$. Finally we obtain $\varepsilon(\beta\,\iota_P) = \gamma\,\iota_P = \gamma_P$. Hence $\beta\,\iota_P : P \to B$ is the desired homomorphism. Thus P is projective; similarly Q is projective. $\square$

In Theorem 4.7 below we shall give a number of different characterizations of projective modules. As a preparation we define:

Definition. A short exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules *splits* if there exists a right inverse to ε, i.e. a homomorphism $\sigma : C \to B$ such that $\varepsilon\sigma = 1_C$. The map σ is then called a *splitting*.

We remark that the sequence $A \overset{\iota_A}{\rightarrowtail} A \oplus C \overset{\pi_C}{\twoheadrightarrow} C$ is exact, and splits by the homomorphism ι_C. The following lemma shows that all split short exact sequences of modules are of this form (see Exercise 3.7).

Lemma 4.6. *Suppose that $\sigma : C \to B$ is a splitting for the short exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$. Then B is isomorphic to the direct sum $A \oplus C$. Under this isomorphism, μ corresponds to ι_A and σ to ι_C.*

In this case we shall say that C (like A) is a *direct summand* in B.

Proof. By the universal property of the direct sum we define a map ψ as follows

$$\begin{tikzcd} A \arrow[dr, "\iota_A"'] \arrow[drr, "\mu"] & & \\ & A \oplus C \arrow[r, dotted, "\psi"] & B \\ C \arrow[ur, "\iota_C"] \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

6.2. Give a universal characterization of kernel and cokernel, and show that kernel and cokernel are dual notions.

6.3. Dualize the assertions of Lemma 1.1, the Five Lemma (Exercise 1.2) and those of Exercises 3.4 and 3.5.

6.4. Let $\varphi: A \to B$. Characterize $\operatorname{im}\varphi$, $\varphi^{-1}B_0$ for $B_0 \subseteq B$, without using elements. What are their duals? Hence (or otherwise) characterize exactness.

6.5. What is the dual of the canonical homomorphism $\sigma: \bigoplus_{i \in J} A_i \to \prod_{i \in J} A_i$? What is the dual of the assertion that σ is an injection? Is the dual true?

7. Injective Modules over a Principal Ideal Domain

Recall that by Corollary 5.2 every projective module over a principal ideal domain is free. It is reasonable to expect that the injective modules over a principal ideal domain also have a simple structure. We first define:

Definition. Let Λ be an integral domain. A Λ-module D is *divisible* if for every $d \in D$ and every $0 \neq \lambda \in \Lambda$ there exists $c \in D$ such that $\lambda c = d$. Note that we do not require the uniqueness of c.

We list a few examples:

(a) As $\mathbb{Z}$-module the additive group of the rationals $\mathbb{Q}$ is divisible. In this example c is uniquely determined.

(b) As $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible. Here c is *not* uniquely determined.

(c) The additive group of the reals $\mathbb{R}$, as well as $\mathbb{R}/\mathbb{Z}$, are divisible.

(d) A non-trivial finitely generated abelian group A is never divisible. Indeed, A is a direct sum of cyclic groups, which clearly are not divisible.

Theorem 7.1. *Let Λ be a principal ideal domain. A Λ-module is injective if and only if it is divisible.*

Proof. First suppose D is injective. Let $d \in D$ and $0 \neq \lambda \in \Lambda$. We have to show that there exists $c \in D$ such that $\lambda c = d$. Define $\alpha: \Lambda \to D$ by $\alpha(1) = d$ and $\mu: \Lambda \to \Lambda$ by $\mu(1) = \lambda$. Since Λ is an integral domain, $\mu(\xi) = \xi\lambda = 0$ if and only if $\xi = 0$. Hence μ is monomorphic. Since D is injective, there exists $\beta: \Lambda \to D$ such that $\beta\mu = \alpha$. We obtain

$$d = \alpha(1) = \beta\mu(1) = \beta(\lambda) = \lambda\beta(1).$$

Hence by setting $c = \beta(1)$ we obtain $d = \lambda c$. (Notice that so far no use is made of the fact that Λ is a principal ideal domain.)

Now suppose D is divisible. Consider the following diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\mu"] \arrow[d, "\alpha"'] & \Lambda \arrow[dl, "\beta"] \\ D & \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, tail, "\mu"] \arrow[d, "\alpha"'] & B \\ D & \end{tikzcd}$$

Is A_j a free module over Λ?

We have to show the existence of $\beta: B \to D$ such that $\beta\mu = \alpha$. To simplify the notation we consider μ as an embedding of a submodule A into B. We look at pairs (A_j, α_j) with $A \subseteq A_j \subseteq B$, $\alpha_j: A_j \to D$ such that $\alpha_j|_A = \alpha$. Let Φ be the set of all such pairs. Clearly Φ is nonempty, since (A, α) is in Φ. The relation $(A_j, \alpha_j) \leqq (A_k, \alpha_k)$ if $A_j \subseteq A_k$ and $\alpha_k|_{A_j} = \alpha_j$ defines an ordering in Φ. With this ordering Φ is inductive. Indeed, every chain (A_j, α_j), $j \in J$ has an upper bound, namely $(\bigcup A_j, \bigcup \alpha_j)$ where $\bigcup A_j$ is simply the union, and $\bigcup \alpha_j$ is defined as follows: If $a \in \bigcup A_j$, then $a \in A_k$ for some $k \in J$. We define $\bigcup \alpha_j(a) = \alpha_k(a)$. Plainly $\bigcup \alpha_j$ is well-defined and is a homomorphism, and

$$(A_j, \alpha_j) \leqq (\bigcup A_j, \bigcup \alpha_j).$$

By Zorn's Lemma there exists a maximal element $(\bar{A}, \bar{\alpha})$ in Φ. We shall show that $\bar{A} = B$, thus proving the theorem. Suppose $\bar{A} \neq B$; then there exists $b \in B$ with $b \notin \bar{A}$. The set of $\lambda \in \Lambda$ such that $\lambda b \in \bar{A}$ is readily seen to be an ideal of Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_0. If $\lambda_0 \neq 0$, then we use the fact that D is divisible to find $c \in D$ such that $\bar{\alpha}(\lambda_0 b) = \lambda_0 c$. If $\lambda_0 = 0$, we choose an arbitrary c. The homomorphism $\bar{\alpha}$ may now be extended to the module $\tilde{A}$ generated by $\bar{A}$ and b, by setting $\tilde{\alpha}(\bar{a} + \lambda b) = \bar{\alpha}(\bar{a}) + \lambda c$. We have to check that this definition is consistent. If $\lambda b \in \bar{A}$, we have $\tilde{\alpha}(\lambda b) = \lambda c$. But $\lambda = \xi \lambda_0$ for some $\xi \in \Lambda$ and therefore $\lambda b = \xi \lambda_0 b$. Hence

$$\bar{\alpha}(\lambda b) = \bar{\alpha}(\xi \lambda_0 b) = \xi \bar{\alpha}(\lambda_0 b) = \xi \lambda_0 c = \lambda c.$$

Since $(\bar{A}, \bar{\alpha}) < (\tilde{A}, \tilde{\alpha})$, this contradicts the maximality of $(\bar{A}, \bar{\alpha})$, so that $\bar{A} = B$ as desired. $\square$

Proposition 7.2. *Every quotient of a divisible module is divisible.*

Proof. Let $\varepsilon: D \twoheadrightarrow E$ be an epimorphism and let D be divisible. For $e \in E$ and $0 \neq \lambda \in \Lambda$ there exists $d \in D$ with $\varepsilon(d) = e$ and $d' \in D$ with $\lambda d' = d$. Setting $e' = \varepsilon(d')$ we have $\lambda e' = \lambda \varepsilon(d') = \varepsilon(\lambda d') = \varepsilon(d) = e$. $\square$

As a corollary we obtain the dual of Corollary 5.3.

Corollary 7.3. *Let Λ be a principal ideal domain. Every quotient of an injective Λ-module is injective.* $\square$

Next we restrict ourselves temporarily to abelian groups and prove in that special case

Proposition 7.4. *Every abelian group may be embedded in a divisible (hence injective) abelian group.*

The reader may compare this Proposition to Proposition 4.3, which says that every Λ-module is a quotient of a free, hence projective, Λ-module.

Proof. We shall define a monomorphism of the abelian group A into a direct product of copies of $\mathbb{Q}/\mathbb{Z}$. By Proposition 6.3 this will suffice. Let $0 \neq a \in A$ and let (a) denote the subgroup of A generated by a. Define $\alpha: (a) \to \mathbb{Q}/\mathbb{Z}$ as follows: If the order of $a \in A$ is infinite choose $0 \neq \alpha(a)$ arbitrary. If the order of $a \in A$ is finite, say n, choose $0 \neq \alpha(a)$ to have order dividing n. Since $\mathbb{Q}/\mathbb{Z}$ is injective, there exists a map $\beta_a: A \to \mathbb{Q}/\mathbb{Z}$ such that the diagram

$$\begin{tikzcd} (a) \arrow[r, tail] \arrow[d, "\alpha"'] & A \arrow[dl, dotted, "\beta_a"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

is commutative. By the universal property of the product, the β_a define a unique homomorphism $\beta: A \to \prod_{\substack{a \in A \\ a \neq 0}} (\mathbb{Q}/\mathbb{Z})_a$. Clearly β is a monomorphism since $\beta_a(a) \neq 0$ if $a \neq 0$. $\square$

For abelian groups, the additive group of the integers $\mathbb{Z}$ is projective and has the property that to any abelian group $G \neq 0$ there exists a nonzero homomorphism $\varphi: \mathbb{Z} \to G$. The group $\mathbb{Q}/\mathbb{Z}$ has the dual properties; it is injective and to any abelian group $G \neq 0$ there is a nonzero homomorphism $\psi: G \to \mathbb{Q}/\mathbb{Z}$. Since a direct sum of copies of $\mathbb{Z}$ is called free, we shall term a direct product of copies of $\mathbb{Q}/\mathbb{Z}$ *cofree*. Note that the two properties of $\mathbb{Z}$ mentioned above do *not* characterize $\mathbb{Z}$ entirely. Therefore "cofree" is *not* the exact dual of "free", it is dual only in certain respects. In Section 8 the generalization of this concept to arbitrary rings is carried through.

Exercises:

7.1. Prove the following proposition: The Λ-module I is injective if and only if for every left ideal $J \subset \Lambda$ and for every Λ-module homomorphism $\alpha: J \to I$ the diagram

$$\begin{tikzcd} J \arrow[r, tail] \arrow[d, "\alpha"'] & \Lambda \arrow[dl, dotted, "\beta"] \\ I & \end{tikzcd}$$

may be completed by a homomorphism $\beta: \Lambda \to I$ such that the resulting triangle is commutative. (Hint: Proceed as in the proof of Theorem 7.1.)

7.2. Let $0 \to R \to F \to A \to 0$ be a short exact sequence of abelian groups, with F free. By embedding F in a direct sum of copies of $\mathbb{Q}$, show how to embed A in a divisible group.

7.3. Show that every abelian group admits a unique maximal divisible subgroup.

7.4. Show that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A$. Deduce that if there is a short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups with A finite, then there is a short exact sequence $0 \to A'' \to A \to A' \to 0$.

7.5. Show that a torsion-free divisible group D is a $\mathbb{Q}$-vector space. Show that $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is then also divisible. Is this true for any divisible group D?

7.6. Show that $\mathbb{Q}$ is a direct summand in a direct product of copies of $\mathbb{Q}/\mathbb{Z}$.

Similarly, if $0 \to A^\cdot \xrightarrow{f} B^\cdot \xrightarrow{g} C^\cdot \to 0$ *is a short exact sequence of cochain complexes, there are natural maps* $\partial\colon H^n(C) \to H^{n+1}(A)$ *and a long exact sequence*

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{f} \cdots.$$

Exercise 1.3.1 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third.

Exercise 1.3.2 (3 × 3 lemma) Suppose given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in an abelian category, such that every column is exact. Show the following:

1. If the bottom two rows are exact, so is the top row.
2. If the top two rows are exact, so is the bottom row.
3. If the top and bottom rows are exact, and the composite $A \to C$ is zero, the middle row is also exact.

Hint: Show the remaining row is a complex, and apply exercise 1.3.1.

The key tool in constructing the connecting homomorphism ∂ is our next result, the *Snake Lemma*. We will not print the proof in these notes, because it is best done visually. In fact, a clear proof is given by Jill Clayburgh at the beginning of the movie *It's My Turn* (Rastar-Martin Elfand Studios, 1980). As an exercise in "diagram chasing" of elements, the student should find a proof (but privately—keep the proof to yourself!).

Snake Lemma 1.3.2 *Consider a commutative diagram of* R*-modules of the form*

$$\begin{array}{ccccccc} & & A' & \longrightarrow & B' & \xrightarrow{p} & C' & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C. & & \end{array}$$

If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

with ∂ defined by the formula

$$\partial(c') = i^{-1} g p^{-1}(c'), \quad c' \in \ker(h).$$

Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, and if $B \to C$ is onto, then so is $\operatorname{coker}(f) \to \operatorname{coker}(g)$.

Etymology The term *snake* comes from the following visual mnemonic:

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] \arrow[dddll, dashed, bend left, to path={}] \\ \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h). \end{tikzcd}$$

Remark The Snake Lemma also holds in an arbitrary abelian category $\mathcal{C}$. To see this, let $\mathcal{A}$ be the smallest abelian subcategory of $\mathcal{C}$ containing the objects and morphisms of the diagram. Since $\mathcal{A}$ has a set of objects, the Freyd-Mitchell Embedding Theorem (see 1.6.1) gives an exact, fully faithful embedding of $\mathcal{A}$ into R–**mod** for some ring R. Since ∂ exists in R–**mod**, it exists in $\mathcal{A}$ and hence in $\mathcal{C}$. Similarly, exactness in R–**mod** implies exactness in $\mathcal{A}$ and hence in $\mathcal{C}$.

Exercise 1.3.3 (5–Lemma) In any commutative diagram

$$\begin{array}{ccccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\ a\downarrow\cong & & b\downarrow\cong & & c\downarrow & & d\downarrow\cong & & e\downarrow\cong \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

with exact rows in any abelian category, show that if a, b, d, and e are isomorphisms, then c is also an isomorphism. More precisely, show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi.

We now proceed to the construction of the connecting homomorphism ∂ of Theorem 1.3.1 associated to a short exact sequence

$$0 \to A \to B \to C \to 0$$

of chain complexes. From the Snake Lemma and the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_nA & \longrightarrow & Z_nB & \longrightarrow & Z_nC & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\ & & d\downarrow & & d\downarrow & & d\downarrow & & \\ 0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \dfrac{A_{n-1}}{dA_n} & \longrightarrow & \dfrac{B_{n-1}}{dB_n} & \longrightarrow & \dfrac{C_{n-1}}{dC_n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

we see that the rows are exact in the commutative diagram

$$\begin{array}{ccccccccc} & & \dfrac{A_n}{dA_{n+1}} & \longrightarrow & \dfrac{B_n}{dB_{n+1}} & \longrightarrow & \dfrac{C_n}{dC_{n+1}} & \longrightarrow & 0 \\ & & d\downarrow & & d\downarrow & & d\downarrow & & \\ 0 & \longrightarrow & Z_{n-1}(A) & \xrightarrow{f} & Z_{n-1}(b) & \xrightarrow{g} & Z_{n-1}(C). & & \end{array}$$

The kernel of the left vertical is $H_n(A)$, and its cokernel is $H_{n-1}(A)$. Therefore the Snake Lemma yields an exact sequence

$$H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

The long exact sequence 1.3.1 is obtained by pasting these sequences together.

Addendum 1.3.3 When one computes with modules, it is useful to be able to push elements around. By decoding the above proof, we obtain the following formula for the connecting homomorphism: Let $z \in H_n(C)$, and represent it by a cycle $c \in C_n$. Lift the cycle to $b \in B_n$ and apply d. The element db of B_{n-1} actually belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$.

We shall now explain what we mean by the naturality of ∂. There is a category $\mathcal{S}$ whose objects are short exact sequences of chain complexes (say, in an abelian category $\mathcal{C}$). Commutative diagrams

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

give the morphisms in $\mathcal{S}$ (from the top row to the bottom row). Similarly, there is a category $\mathcal{L}$ of long exact sequences in $\mathcal{C}$.

Proposition 1.3.4 *The long exact sequence is a functor from $\mathcal{S}$ to $\mathcal{L}$. That is, for every short exact sequence there is a long exact sequence, and for every map $(*)$ of short exact sequences there is a commutative ladder diagram*

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial} & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & H_n(A') & \longrightarrow & H_n(B') & \longrightarrow & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \longrightarrow & \cdots . \end{array}$$

Proof All we have to do is establish the ladder diagram. Since each H_n is a functor, the left two squares commute. Using the Embedding Theorem 1.6.1, we may assume $\mathcal{C} = \mathbf{mod}\text{–}R$ in order to prove that the right square commutes. Given $z \in H_n(C)$, represented by $c \in C_n$, its image $z' \in H_n(C')$ is represented by the image of c. If $b \in B_n$ lifts c, its image in B'_n lifts c'. Therefore by 1.3.3 $\partial(z') \in H_{n-1}(A')$ is represented by the image of db, that is, by the image of a representative of $\partial(z)$, so $\partial(z')$ is the image of $\partial(z)$. ◇

$(-)_G \overset{\perp}{\rightleftarrows} (-)^G$

preserve colimits

preserve limits

6

Group Homology and Cohomology

G-mod is a module obtained by action of a group G on an abelian group. It can be identified w/ $\mathbb{Z}G$ or Ab^G.

6.1 Definitions and First Properties

Let G be a group. A (left) *G-module* is an abelian group A on which G acts by additive maps on the left; if $g \in G$ and $a \in A$, we write ga for the action of g on a. Letting $\mathrm{Hom}_G(A, B)$ denote the G-set maps from A to B, we obtain a category G–**mod** of left G-modules. The category G–**mod** may be identified with the category $\mathbb{Z}G$–**mod** of left modules over the integral group ring $\mathbb{Z}G$. It may also be identified with the functor category $\mathbf{Ab}^G$ of functors from the category "G" (one object, G being its endomorphisms) to the category **Ab** of abelian groups.

A *trivial G-module* is an abelian group A on which G acts "trivially," that is, $ga = a$ for all $g \in G$ and $a \in A$. Considering an abelian group as a trivial G-module provides an exact functor from **Ab** to G–**mod**. Consider the following two functors from G–**mod** to **Ab**:

1. The *invariant subgroup* A^G of a G-module A,
$$A^G = \{a \in A : ga = a \text{ for all } g \in G \text{ and } a \in A\}.$$
2. The *coinvariants* A_G of a G-module A,
$$A_G = A/\text{submodule generated by } \{(ga - a) : g \in G, a \in A\}.$$

A^G — invariant subgroup $(ga = a)$

A_G — covariants $(ga - a)$

Exercise 6.1.1

1. Show that A^G is the maximal trivial submodule of A, and conclude that the invariant subgroup functor $-^G$ is right adjoint to the trivial module functor. Conclude that $-^G$ is a left exact functor.

$Ab \longrightarrow G\text{-mod}$: trivial, exact

$G\text{-mod} \longrightarrow Ab$: two functors

$(-)_G \overset{\perp}{\rightleftarrows} Ab \longrightarrow G\text{-Mod}$
$\overset{\perp}{\rightleftarrows} (-)^G$

2. Show that A_G is the largest quotient module of A that is trivial, and conclude that the coinvariants functor $-_G$ is left adjoint to the trivial module functor. Conclude that $-_G$ is a right exact functor.

$A \otimes - \overset{\perp}{\rightleftarrows} \mathrm{Hom}(A, -)$

Lemma 6.1.1 *Let A be any G-module, and let $\mathbb{Z}$ be the trivial G-module. Then $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A^G \cong \mathrm{Hom}_G(\mathbb{Z}, A)$.*

Proof Considering $\mathbb{Z}$ as a $\mathbb{Z}$–$\mathbb{Z}G$ bimodule, the "trivial module functor" from $\mathbb{Z}$–**mod** to $\mathbb{Z}G$–**mod** is the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$. We saw in 2.6.3 that $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ is its left adjoint; this functor must agree with its other left adjoint $(-)_G$. For the second equation, we use adjointness: $A^G \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, A^G) \cong \mathrm{Hom}_G(\mathbb{Z}, A)$. ◇

Definition 6.1.2 Let A be a G-module. We write $H_*(G; A)$ for the left derived functors $L_*(-_G)(A)$ and call them the *homology groups of G with coefficients in A*; by the lemma above, $H_*(G; A) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$. By definition, $H_0(G; A) = A_G$. Similarly, we write $H^*(G; A)$ for the right derived functors $R^*(-^G)(A)$ and call them the *cohomology groups of G with coefficients in A*; by the lemma above, $H^*(G; A) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A)$. By definition, $H^0(G; A) = A^G$.

Example 6.1.3 If $G = 1$ is the trivial group, $A_G = A^G = A$. Since the higher derived functors of an exact functor vanish, $H_*(1; A) = H^*(1; A) = 0$ for $* \neq 0$.

Example 6.1.4 Let G be the infinite cyclic group T with generator t. We may identify $\mathbb{Z}T$ with the Laurent polynomial ring $\mathbb{Z}[t, t^{-1}]$. Since the sequence

$$0 \to \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \to \mathbb{Z} \to 0$$

is exact,

$$H_n(T; A) = H^n(T; A) = 0 \text{ for } n \neq 0, 1, \text{ and}$$
$$H_1(T; A) \cong H^0(T; A) = A^T,\ H^1(T; A) \cong H_0(T; A) = A_T.$$

In particular, $H_1(T; \mathbb{Z}) = H^1(T; \mathbb{Z}) = \mathbb{Z}$. We will see in the next section that all free groups display similar behavior, because $pd_G(\mathbb{Z}) = 1$.

Exercise 6.1.2 (kG-modules) As a variation, we can replace $\mathbb{Z}$ by any commutative ring k and consider the category kG–**mod** of k-modules on which G acts k-linearly. The functors A_G and A^G from kG–**mod** to k–**mod** are left

$A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A,\ A^G = \mathrm{Hom}_G(\mathbb{Z}, A)$

(resp. right) exact and may be used to form the derived functors Tor_*^{kG} and Ext_{kG}^*. Prove that if A is a kG-module, then we have isomorphisms of abelian groups

$$H_*(G;A)\cong \mathrm{Tor}_*^{kG}(k,A) \quad \text{and} \quad H^*(G;A)\cong \mathrm{Ext}_{kG}^*(k,A).$$

This proves that $H_*(G;A)$ and $H^*(G;A)$ are k-modules whenever A is a kG-module. *Hint:* If $P\to\mathbb{Z}$ is a projective $\mathbb{Z}G$-resolution, consider $P\otimes_{\mathbb{Z}} k\to k$.

We now return our attention to H_0 and H^0.

augmentation ideal

Definition 6.1.5 The *augmentation ideal* of $\mathbb{Z}G$ is the kernel $\mathfrak{I}$ of the ring map $\mathbb{Z}G\xrightarrow{\epsilon}\mathbb{Z}$ which sends $\sum n_g g$ to $\sum n_g$. Because $\{1\}\cup\{g-1 : g\in G, g\neq 1\}$ is a basis for $\mathbb{Z}G$ as a free $\mathbb{Z}$-module, it follows that $\mathfrak{I}$ is a free $\mathbb{Z}$-module with basis $\{g-1 : g\in G, g\neq 1\}$.

IG

Example 6.1.6 Since the trivial G-module $\mathbb{Z}$ is $\mathbb{Z}G/\mathfrak{I}$, $H_0(G;A)=A_G$ is isomorphic to $\mathbb{Z}\otimes_{\mathbb{Z}G}A=\mathbb{Z}G/\mathfrak{I}\otimes_{\mathbb{Z}G}A\cong A/\mathfrak{I}A$ for every G-module A. For example, $H_0(G;\mathbb{Z})=\mathbb{Z}/\mathfrak{I}\mathbb{Z}=\mathbb{Z}$, $H_0(G;\mathbb{Z}G)=\mathbb{Z}G/\mathfrak{I}\cong\mathbb{Z}$, and $H_0(G;\mathfrak{I})=\mathfrak{I}/\mathfrak{I}^2$.

$H_0(G,A) = A_G$

Example 6.1.7 $(A=\mathbb{Z}G)$ Because $\mathbb{Z}G$ is a projective object in $\mathbb{Z}G$–**mod**, $H_*(G;\mathbb{Z}G)=0$ for $*\neq 0$ and $H_0(G;\mathbb{Z}G)=\mathbb{Z}$. When G is a finite group, Shapiro's Lemma (6.3.2 below) implies that $H^*(G;\mathbb{Z}G)=0$ for $*\neq 0$. This fails when G is infinite; for example, we saw in 6.1.4 that $H^1(T;\mathbb{Z}T)\cong\mathbb{Z}$ for the infinite cyclic group T.

$H^0(G,A) = A^G$

The following discussion clarifies the situation for $H^0(G;\mathbb{Z}G)$: If G is finite, then $H^0(G;\mathbb{Z}G)\cong\mathbb{Z}$, but $H^0(G;\mathbb{Z}G)=0$ if G is infinite.

The Norm Element 6.1.8 Let G be a finite group. The *norm element* N of the group ring $\mathbb{Z}G$ is the sum $N=\sum_{g\in G} g$. The norm is a central element of $\mathbb{Z}G$ and belongs to $(\mathbb{Z}G)^G$, because for every $h\in G$ $hN=\sum_g hg=\sum_{g'} g'=N$, and $Nh=N$ similarly.

Lemma 6.1.9 *The subgroup $H^0(G;\mathbb{Z}G)=(\mathbb{Z}G)^G$ of $\mathbb{Z}G$ is the 2-sided ideal $\mathbb{Z}\cdot N$ of $\mathbb{Z}G$ (isomorphic to $\mathbb{Z}$) generated by N.*

Proof If $a=\sum n_g g$ is in $(\mathbb{Z}G)^G$, then $a=ga$ for all $g\in G$. Comparing coefficients of g shows that all the n_g are the same. Hence $a=nN$ for some $n\in\mathbb{Z}$. ◇

$\mathbb{Z} \to \mathscr{C}$

$\mathrm{mor}(\mathscr{C})$: Mod $(\mathbb{Z})$

Hence, the category $\mathrm{Diff}(\mathscr{C})$ of differential objects in $\mathscr{C}$ is nothing but the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$. In particular, it is an additive category.

Definition 3.2.1. (i) A complex is a differential object $(X^\bullet, d_X^\bullet)$ such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

(ii) One denotes by $\mathrm{C}(\mathscr{C})$ the full additive subcategory of $\mathrm{Diff}(\mathscr{C})$ consisting of complexes.

$C(\mathscr{C})$ is a full subcat. of $\mathscr{C}$.

From now on, we shall concentrate our study on the category $\mathrm{C}(\mathscr{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathscr{C})(* = \mathrm{b}, +, -)$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of bounded complexes (resp. bounded below, bounded above). We also use the notation $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) = \mathrm{C}(\mathscr{C})$ (ub for "unbounded"). For $a \in \mathbb{Z}$ we shall denote by $\mathrm{C}^{\geq a}(\mathscr{C})$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects $X^\bullet$ such that $X^j \simeq 0$ for $j < a$. One defines similarly the categories $\mathrm{C}^{\leq a}(\mathscr{C})$ and, for $a \leq b$, $\mathrm{C}^{[a,b]}(\mathscr{C})$.

One considers $\mathscr{C}$ as a full subcategory of $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ by identifying an object $X \in \mathscr{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0. In other words, one identifies $\mathscr{C}$ and $\mathrm{C}^{[0,0]}(\mathscr{C})$.

Shift functor

Let $\mathscr{C}$ be an additive category, let $X \in \mathrm{C}(\mathscr{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by:

$$(X[p])^n = X^{n+p}$$
$$d^n_{X[p]} = (-1)^p d_X^{n+p}$$

$$\begin{tikzcd} Y \arrow[r] & X[1] \oplus Y \arrow[d] \\ & X[1] \end{tikzcd}$$

If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$ one defines $f[p]\colon X[p] \to Y[p]$ by $(f[p])^n = f^{n+p}$.

The shift functor $[1]\colon X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathscr{C})$.

Mapping cone

Definition 3.2.2. Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathscr{C})$ defined by:

$$\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$$
$$d^n_{\mathrm{Mc}(f)} = \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \mathrm{cone}(X) \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

n

Of course, before to state this definition, one should check that $d^{n+1}_{\mathrm{Mc}(f)} \circ d^n_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d_X^{n+2} & 0 \\ f^{n+2} & d_Y^{n+1} \end{pmatrix} \circ \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = 0$$

$$\begin{tikzcd} X^n \arrow[r] \arrow[dr] & X^{n+1} \arrow[r, "d^n_{X[1]}"] \arrow[dr, "f^{n+1}"] & X^{n+2} \\ Y^{n-1} \arrow[r] & Y^n \arrow[r, "d^n_Y"] & Y^{n+1} \end{tikzcd}$$

Notice that although $\mathrm{Mc}(f)^n = (X[1])^n \oplus Y^n$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $\mathrm{C}(\mathscr{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

(3.2.2) $$\alpha(f)\colon Y \to \mathrm{Mc}(f), \quad \beta(f)\colon \mathrm{Mc}(f) \to X[1].$$

* split exact sequence

and $\beta(f) \circ \alpha(f) = 0$.

If $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

The homotopy category $\mathrm{K}(\mathscr{C})$

Let again $\mathscr{C}$ be an additive category.

Definition 3.2.3. (i) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is homotopic to zero if for all p there exists a morphism $s^p\colon X^p \to Y^{p-1}$ such that:

$$f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathscr{C})$ is homotopic to 0 if id_X is homotopic to zero.

(iii) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

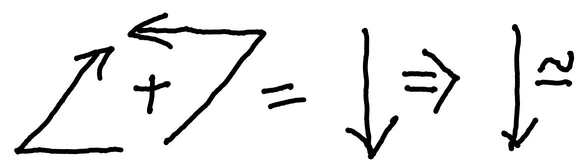
hom-otopic cxs.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\
Y
\end{tikzcd}$$

$f - g = 0 \Rightarrow f \simeq g$

$$\begin{tikzcd}
X^{p-1} \arrow[r] & X^p \arrow[r, "d_X^p"] \arrow[dl, "s^p"'] \arrow[d, "f^p"] & X^{p+1} \arrow[dl, "s^{p+1}"'] \\
Y^{p-1} \arrow[r, "d_Y^{p-1}"'] & Y^p \arrow[r] & Y^{p+1}.
\end{tikzcd}$$



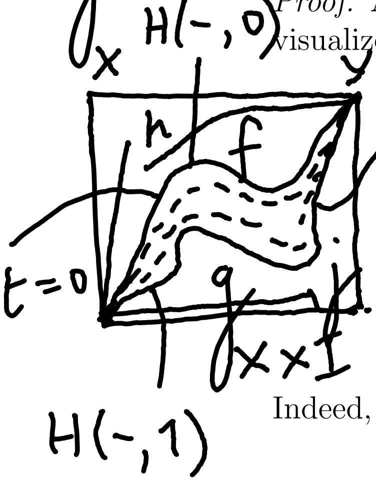
Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example 3.2.4. The complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero. *

Lemma 3.2.5. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms in $\mathrm{C}(\mathscr{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero.*

$f, g : X \to Y$

Proof. Assume for example the f is homotopic to zero. In this case the proof is visualized by the diagram below.



$f \simeq g \neq j \neq h$

$$\begin{tikzcd}
X^{p-1} \arrow[r] & X^p \arrow[r, "d_X^p"] \arrow[dl, "s^p"'] \arrow[d, "f^p"] & X^{p+1} \arrow[dl, "s^{p+1}"'] \\
Y^{p-1} \arrow[r] \arrow[d, "g^{p-1}"] & Y^p \arrow[r] \arrow[d, "g^p"] & Y^{p+1} \arrow[d, "g^{p+1}"] \\
Z^{p-1} \arrow[r, "d_Z^{p-1}"'] & Z^p \arrow[r] & Z^{p+1}
\end{tikzcd}$$

homotopic chains

Indeed, the equality $f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p$ implies

$$g^p \circ f^p = g^p \circ s^{p+1} \circ d_X^p + d_Z^{p-1} \circ g^{p-1} \circ s^p.$$

□

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[dr, bend right, "f"'] & X \times I \arrow[d, "H"] & X \arrow[l, "j"'] \arrow[dl, bend left, "g"] \\
& Y &
\end{tikzcd}$$

$$\left.\begin{aligned} H(-,0) &= f \\ H(-,1) &= g \end{aligned}\right\} \quad H(x,t) = y$$

We shall construct a new category by deciding that a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X,Y) = \{f\colon X \to Y; f \text{ is homotopic to } 0\}.$$

Lemma 3.2.5 allows us to state:

w = quasi-iso
weak equivs.
become isos.

Definition 3.2.6. The homotopy category $\mathrm{K}(\mathscr{C})$ is defined by:

K(C) — homotopy cat. of C

$$\mathrm{Ob}(\mathrm{K}(\mathscr{C})) = \mathrm{Ob}(\mathrm{C}(\mathscr{C}))$$
$$\mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X,Y) = \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y)/Ht(X,Y).$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathscr{C})$ becomes the zero morphism in $\mathrm{K}(\mathscr{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathscr{C})$, $(* = \mathrm{ub}, \mathrm{b}, +, -)$. They are clearly additive categories endowed with an automorphism, the shift functor $[1]\colon X \mapsto X[1]$.

3.3 Double complexes

Let $\mathscr{C}$ be as above an additive category. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathscr{C}$ is the data of

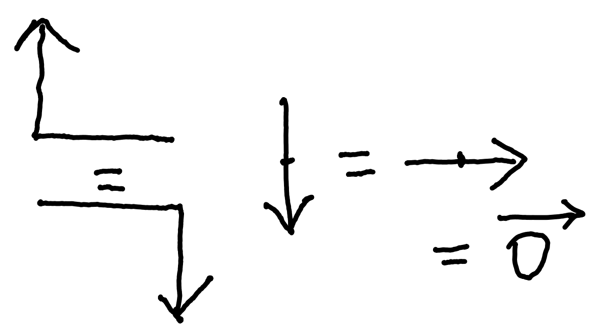
$$\{X^{n,m}, d'^{n,m}_X, d''^{n,m}_X; (n,m) \in \mathbb{Z}\times\mathbb{Z}\}$$

where $X^{n,m} \in \mathscr{C}$ and the "differentials" $d'^{n,m}_X\colon X^{n,m} \to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m} \to X^{n,m+1}$ satisfy:

$$d'^2_X = d''^2_X = 0, \quad d' \circ d'' = d'' \circ d'. \tag{3.3.1}$$

One can represent a double complex by a commutative diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & X^{n,m} \arrow[r, "d''^{n,m}"] \arrow[d, "d'^{n,m}"'] & X^{n,m+1} \arrow[r] \arrow[d, "d'^{n,m+1}"'] & {} \\
{} \arrow[r] & X^{n+1,m} \arrow[r, "d''^{n+1,m}"'] \arrow[d] & X^{n+1,m+1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd} \tag{3.3.2}$$



One defines naturally the notion of a morphism of double complexes and one obtains the additive category $\mathrm{C}^2(\mathscr{C})$ of double complexes.

There are two functors $F_I, F_{II} : \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$ which associate to a double complex X the complex whose objects are the rows (resp. the columns) of X. These two functors are clearly isomorphisms of categories.

Now consider the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z}\times\mathbb{Z}; X^{n,m} \neq 0, m+n = p\} \text{ is finite} \tag{3.3.3}$$

[1]§ 3.3 may be skipped in a first reading.

and denote by $C^2_f(\mathscr{C})$ the full subcategory of $\mathrm{C}^2(\mathscr{C})$ consisting of objects X satisfying (3.3.3). To such an X one associates its "total complex" $\mathrm{tot}(X)$ by setting:

$$\begin{aligned} \mathrm{tot}(X)^p &= \oplus_{m+n=p} X^{n,m}, \\ d^p_{\mathrm{tot}(X)}|_{X^{n,m}} &= d'^{n,m} + (-1)^n d''^{n,m}. \end{aligned}$$

This is visualized by the diagram:

$$\begin{tikzcd} X^{n,m} \arrow[r, "(-)^n d''"] \arrow[d, "d'"'] & X^{n,m+1} \\ X^{n+1,m} & \end{tikzcd}$$

$d \circ d = d''d'' + d'd'$
$+ (-1)^n d''d'$
$+ (-1)^n d'd''$

Proposition 3.3.1. *The differential object $\{\mathrm{tot}(X)^p, d^p_{\mathrm{tot}(X)}\}_{p\in\mathbb{Z}}$ is a complex* (i.e., $d^{p+1}_{\mathrm{tot}(X)} \circ d^p_{\mathrm{tot}(X)} = 0$) *and* $\mathrm{tot}\colon C^2_f(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ *is a functor of additive categories.*

Proof. For $(n,m) \in \mathbb{Z}\times\mathbb{Z}$, one has

$$\begin{aligned} d\circ d(X^{n,m}) &= d''\circ d''(X^{n,m}) + d'\circ d'(X^{n,m}) \\ &\quad + (-)^{n\cancel{+1}} d''\circ d'(X^{n,m}) + (-)^n d'\circ d''(X^{n,m}) \\ &= 0. \end{aligned}$$

It is left to the reader to check that tot is an additive functor. □

Example 3.3.2. Let $f^\bullet\colon X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathscr{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j\colon Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{3.3.4}$$

Bifunctor

Let $\mathscr{C}, \mathscr{C}'$ and $\mathscr{C}''$ be additive categories and let $F\colon \mathscr{C}\times\mathscr{C}' \to \mathscr{C}''$ be an additive bifunctor (*i.e.*, $F(\bullet,\bullet)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F)\colon \mathrm{C}(\mathscr{C})\times\mathrm{C}(\mathscr{C}') \to \mathrm{C}^2(\mathscr{C}'')$. In other words, if $X\in\mathrm{C}(\mathscr{C})$ and $X'\in\mathrm{C}(\mathscr{C}')$ are complexes, then $\mathrm{C}^2(F)(X,X')$ is a double complex.

Example 3.3.3. Consider the bifunctor $\bullet\otimes\bullet\colon \mathrm{Mod}(A^{\mathrm{op}})\times\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Then, for $X\in\mathrm{C}^-(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y\in\mathrm{C}^-(\mathrm{Mod}(A))$, one has

$$\begin{aligned} &(X\otimes Y)^{n,m} = X^n\otimes Y^m, \\ &d'^{n,m} = d^n_X\otimes Y^m,\ d''^{n,m} = X^n\otimes d^m_Y. \end{aligned}$$

The complex $\mathrm{Hom}^\bullet$

Consider the bifunctor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall write $\mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathscr{C}})$. If X and Y are two objects of $\mathrm{C}(\mathscr{C})$, one has

$$\begin{aligned} \mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}(X,Y)^{n,m} &= \mathrm{Hom}_{\mathscr{C}}(X^{-m},Y^n), \\ d'^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m},d^n_Y), \quad d''^{m,n} &= \mathrm{Hom}_{\mathscr{C}}((-)^m d^{-m-1}_X, Y^n). \end{aligned}$$

(iii) Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^{\mathrm{f}}(A)$ has no kernel.

Definition 4.1.2. Let $\mathscr{C}$ be an additive category. One says that $\mathscr{C}$ is abelian if:

coim $\cong$ im

(i) any $f\colon X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathscr{C}$, the natural morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

Examples 4.1.3. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.
(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 4.1.1 (ii).)
(iii) If $\mathscr{C}$ is abelian, then $\mathscr{C}^{\mathrm{op}}$ is abelian.

$I \to \mathscr{C}$ is an ab. func. given $\mathscr{C}$ ab.

Proposition 4.1.4. *Let I be category and let $\mathscr{C}$ be an abelian category. Then the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$ is abelian.*

Proof. (i) Let $F, G\colon I \to \mathscr{C}$ be two functors and $\varphi\colon F \to G$ a morphism of functors. Let us define a new functor H as follows. For $i \in I$, set $H(i) = \ker(F(i) \to G(i))$. Let $s\colon i \to j$ be a morphism in I. In order to define the morphism $H(s)\colon H(i) \to H(j)$, consider the diagram

$$\begin{tikzcd}
H(i) \arrow[r, "h_i"] \arrow[d, dotted, "H(s)"'] & F(i) \arrow[r, "\varphi(i)"] \arrow[d, "F(s)"'] & G(i) \arrow[d, "G(s)"] \\
H(j) \arrow[r, "h_j"] & F(j) \arrow[r, "\varphi(i)"] & G(j).
\end{tikzcd}$$

Since $\varphi(j) \circ F(s) \circ h_i = 0$, the morphism $F(s) \circ h_i$ factorizes uniquely through $H(j)$. This gives $H(s)$. One checks immediately that for a morphism $t\colon j \to k$ in I, one has $H(t) \circ H(s) = H(t \circ s)$. Therefore H is a functor and one also easily cheks that H is a kernel of the morphism of functors φ.
(ii) One defines similarly the functor $\mathrm{Coim}\,\varphi$. Since, for each $i \in I$, the natural morphism $\mathrm{Coim}\,\varphi(i) \to \mathrm{Im}\,\varphi(i)$ is an isomorphism, one deduces that the natural morphism of functors $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. $\square$

$C^*(\mathscr{C})$ is abelian

Corollary 4.1.5. *If $\mathscr{C}$ is abelian, then the categories of complexes $\mathrm{C}^*(\mathscr{C})$ $(* = \mathrm{ub}, \mathrm{b}, +, -)$ are abelian.*

Proof. It follows from Proposition 4.1.4 that the category $\mathrm{Diff}(\mathscr{C})$ of differential objects of $\mathscr{C}$ is abelian. One checks immediately that if $f^\bullet\colon X^\bullet \to Y^\bullet$ is a morphism of complexes, its kernel in the category $\mathrm{Diff}(\mathscr{C})$ is a complex and is a kernel in the category $\mathrm{C}(\mathscr{C})$, and similarly with cokernels. $\square$

For example, if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, the complex Z defined by $Z^n = \ker(f^n\colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Note the following results.

- An abelian category admits finite limits and finite colimits. (Indeed, an abelian category admits an initial object, a terminal object, finite products and finite coproducts and kernels and cokernels.)

This condition is equiv. to every arrow admitting ker & coker.

Ker $f = 0 \Rightarrow f$ monic
coker $f = 0 \Rightarrow f$ epic

- In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\ker f \simeq 0$ (resp. $\operatorname{Coker} f \simeq 0$) (see Exercise 2.12). Moreover, a morphism $f\colon X \to Y$ is an isomorphism as soon as $\ker f \simeq 0$ and $\operatorname{Coker} f \simeq 0$. Indeed, in such a case, $X \xrightarrow{\sim} \operatorname{Coim} f$ and $\operatorname{Im} f \xrightarrow{\sim} Y$.

Unless otherwise specified, we assume until the end of this chapter that $\mathscr{C}$ is abelian.

Consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ (hence, $g \circ f = 0$). It defines a morphism $\operatorname{Coim} f \to \ker g$, hence, $\mathscr{C}$ being abelian, a morphism $\operatorname{Im} f \to \ker g$.

Definition 4.1.6. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\operatorname{Im} f \xrightarrow{\sim} \ker g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \ker d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f\colon X \to Y$ may be decomposed into short exact sequences:

$$0 \to \ker f \to X \to \operatorname{Coim} f \to 0,$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0,$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 4.1.7. *Let*

split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, "f"] & X \arrow[r, "g"] \arrow[l, bend left, "k"] & X'' \arrow[r] \arrow[l, bend right, "h"'] & 0 \end{tikzcd} \tag{4.1.2}$$

be a short exact sequence in $\mathscr{C}$. Then the conditions (a) *to* (e) *are equivalent.*

(a) *there exists $h\colon X'' \to X$ such that $g \circ h = \mathrm{id}_{X''}$.*

(b) *there exists $k\colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$.*

(c) *there exists $\varphi = (k, g)$ and $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ such that $X \xrightarrow{\varphi} X' \oplus X''$ and $X' \oplus X'' \xrightarrow{\psi} X$ are isomorphisms inverse to each other.*

(d) *The complex* (4.1.2) *is homotopic to* 0.

(e) *The complex* (4.1.2) *is isomorphic to the complex* $0 \to X' \to X' \oplus X'' \to X'' \to 0$.

Proof. (a) ⇒ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_X - h \circ g) = 0$, which implies that $\mathrm{id}_X - h \circ g$ factors through $\ker g$, that is, through X'. Hence, there exists $k\colon X \to X'$ such that $\mathrm{id}_X - h \circ g = f \circ k$.
(b) ⇒ (c) follows by reversing the arrows.
(c) ⇒ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \mathrm{id}_{X''} = 0$.
(c) ⇒ (b) follows by reversing the arrows.

(d) By definition, the complex (4.1.2) is homotopic to zero if and only if there exists a diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] \arrow[dl, "k"'] & X'' \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"'] & X \arrow[r, "g"'] & X'' \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{id}_{X'} = k \circ f$, $\mathrm{id}_{X''} = g \circ h$ and $\mathrm{id}_X = h \circ g + f \circ k$.
(e) is obvious by (c). ☐

Definition 4.1.8. In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

In a field, all s.e. sequences split.

4.2 Exact functors

Definition 4.2.1. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor of abelian categories. One says that

(i) F is left exact if it commutes with finite limits,

(ii) F is right exact if it commutes with finite colimits,

(iii) F is exact if it is both left and right exact.

Lemma 4.2.2. *Consider an additive functor $F\colon \mathscr{C} \to \mathscr{C}'$.*

(a) *The conditions below are equivalent:*

  (i) *F is left exact,*

  (ii) *F commutes with kernels, that is, for any morphism $f\colon X \to Y$, $F(\ker(f)) \xrightarrow{\sim} \ker(F(f))$,*

  (iii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

  (iv) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$.*

(b) *The conditions below are equivalent:*

  (i) *F is exact,*

  (ii) *for any exact sequence $X' \to X \to X''$ in $\mathscr{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*

 $\bigoplus : I \longrightarrow \mathrm{Mod}(A)$ is exact

(iii) *for any exact sequence* $0 \to X' \to X \to X'' \to 0$ *in* $\mathscr{C}$, *the sequence* $0 \to F(X') \to F(X) \to F(X'') \to 0$ *is exact in* $\mathscr{C}'$.

There is a similar result to (a) for right exact functors.

Proof. Since F is additive, it commutes with terminal objects and products of two objects. Hence, by Proposition 2.3.8, F is left exact if and only if it commutes with kernels.
The proof of the other assertions are left as an exercise. □

Proposition 4.2.3. (i) *The functor* $\mathrm{Hom}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$ *is left exact with respect to each of its arguments.*

(ii) *If a functor* $F : \mathscr{C} \to \mathscr{C}'$ *admits a left (resp. right) adjoint then* F *is left (resp. right) exact.*

(iii) *Let* I *be a small category. If* $\mathscr{C}$ *admits limits indexed by* I, *then the functor* $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ *is left exact. Similarly, if* $\mathscr{C}$ *admits colimits indexed by* I, *then the functor* $\mathrm{colim} : \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$ *is right exact.*

(iv) *Let* A *be a ring and let* I *be a set. The two functors* $\prod_{i \in I}$ *and* $\bigoplus_{i \in I}$ *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *are exact.*

(v) *Let* A *be a ring and let* I *be a small filtrant category. The functor* colim *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *is exact.*

Proof. (i) follows from (2.3.2) and (2.3.3).
(ii) Apply Proposition 2.4.5.
(iii) Apply Proposition 2.4.1.
(iv) is left as an exercise (see Exercise 4.1).
(v) follows from Corollary 2.5.7. □

Example 4.2.4. Let A be a ring and let N be a right A-module. Since the functor $N \otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\bullet, \bullet)$ and $N \otimes_A \bullet$ are not exact in general. In the sequel, we choose $A = \mathbf{k}[x]$, with $\mathbf{k}$ a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{4.2.1}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\bullet, A)$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\bullet, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.

left adjoints preserve colim
right adjoints preserve lim

(iii) Apply $\bullet \otimes_A A/Ax$ to the exact sequence (4.2.1). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.
(iv) Notice that the functor $\mathrm{Hom}_A(\bullet, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (4.2.1) does not split.

Example 4.2.5. We shall show that the functor $\varprojlim : \mathrm{Fct}(I^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k})$ is not right exact in general, even if $\mathbf{k}$ is a field.

Consider as above the $\mathbf{k}$-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn} \colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. One checks easily that

$$\varprojlim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\varprojlim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the (projective) limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact, neither in the category $\mathrm{Mod}(A)$ nor in the category $\mathrm{Mod}(\mathbf{k})$.

The Mittag-Leffler condition

Let us give a criterion in order that the limit of an exact sequence remains exact in the category $\mathrm{Mod}(A)$. This is a particular case of the so-called "Mittag-Leffler" condition (see [Gro61]).

Proposition 4.2.6. *Let A be a ring and let $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ be an exact sequence of projective systems of A-modules indexed by $\mathbb{N}$. Assume that for each n, the map $M'_{n+1} \to M'_n$ is surjective. Then the sequence*

$$0 \to \varprojlim_n M'_n \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M''_n \to 0$$

is exact.

taking lim of
Mittag-Lefler condition: each term in
exact seq. preserves.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p. Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n \colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. □

Hom$_{\mathscr{C}}(\cdot, I)$ and Hom$_{\mathscr{C}}(P, \cdot)$ are exact where I is an inj. and P is a projective

4.3 Injective and projective objects

Definition 4.3.1. Let $\mathscr{C}$ be an abelian category.

(i) An object I of $\mathscr{C}$ is injective if the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ is exact.

(ii) One says that $\mathscr{C}$ has enough injectives if for any $X \in \mathscr{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathscr{C}$ if it is injective in $\mathscr{C}^{\mathrm{op}}$, *i.e.*, if the functor $\mathrm{Hom}_{\mathscr{C}}(P, \,\bullet\,)$ is exact.

(iv) One says that $\mathscr{C}$ has enough projectives if for any $X \in \mathscr{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Proposition 4.3.2. *The object $I \in \mathscr{C}$ is injective if and only if, for any $X, \cancel{Y}\, X' \in \mathscr{C}$ and any diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r, hook, "f"] \arrow[d, "k"'] & X \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd} \quad \exists!$$

the dotted arrow may be completed, making the solid diagram commutative.

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] \arrow[d] & R \arrow[dl, dashed, "\exists!"] \\ & I & \end{tikzcd}$$

Proof. (i) Assume that I is injective and let X'' denote the cokernel of the morphism $X' \to X$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ to the sequence $0 \to X' \to X \to X''$, one gets the exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(X'', I) \to \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0.$$

Thus there exists $h \colon X \to I$ such that $h \circ f = k$.

(ii) Conversely, consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Then the sequence $0 \to \mathrm{Hom}_{\mathscr{C}}(X'', I) \xrightarrow{\circ h} \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0$ is exact by the hypothesis.

To conclude, apply Lemma 4.2.2. □

By reversing the arrows, we get that P is projective if and only if for any diagram in which the row is exact:

$$\begin{tikzcd} & P \arrow[d, "k"] \arrow[dl, dotted, "h"'] & \\ X \arrow[r, two heads, "f"] & X'' \arrow[r] & 0 \end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

free mod: $M \cong \bigoplus_{i \in I} A_i$

$\mathrm{Hom}_{\mathcal{C}}(P, \cdot)$ is exact

Lemma 4.3.3. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h \colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 4.1.7. □

It follows that if $F \colon \mathscr{C} \to \mathscr{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 4.3.4. *Let X', X'' belong to $\mathscr{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, the functor $F \oplus G \colon X \mapsto F(X) \oplus G(X)$ is exact if and only if the functors F and G are exact. □

Applying Lemmas 4.3.3 and 4.3.4, we get:

Proposition 4.3.5. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathscr{C}$ and assume X' and X are injective. Then X'' is injective.*

Example 4.3.6. (i) Let A be a ring. An A-module M is free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{(I)}$ for some small set I. It follows from (2.1.4) and Proposition 4.2.3 (iv) that free modules are projective.

Let $M \in \mathrm{Mod}(A)$. For $m \in M$, denote by A_m a copy of A and denote by $1_m \in A_m$ the unit. Define the linear map

$$\psi \colon \bigoplus_{m \in M} A_m \to M$$

by setting $\psi(1_m) = m$ and extending by linearity. This map is clearly surjective. Since the left A-module $\bigoplus_{m \in M} A_m$ is free, it is projective. This shows that the category $\mathrm{Mod}(A)$ has enough projectives.

More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 4.3).

One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 4.4).

(ii) If $\mathbf{k}$ is a field, then any object of $\mathrm{Mod}(\mathbf{k})$ is both injective and projective.

(iii) Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A^{\mathrm{op}})$. One says that M is flat if the functor $M \otimes_A \bullet \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is exact. Clearly, projective modules are flat.

4.4 Generators and Grothendieck categories

In this section it is essential to fix a universe $\mathscr{U}$. Hence, a category means a $\mathscr{U}$-category and small means $\mathscr{U}$-small.

Definition 4.4.1. Let $\mathscr{C}$ be a category. A system of generators in $\mathscr{C}$ is a family of objects $\{G_i\}_{i \in I}$ of $\mathscr{C}$ such that I is small and a morphism $f \colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $\mathrm{Hom}_{\mathscr{C}}(G_i, X) \to \mathrm{Hom}_{\mathscr{C}}(G_i, Y)$ is an isomorphism for all $i \in I$.

Hom preserves limits: $\prod, \coprod$ are limits

$\mathrm{Hom}_{\mathcal{C}}(\coprod_i A_i, M) = \prod_i \mathrm{Hom}_{\mathcal{C}}(A_i, M)$; $\bigoplus_i A_i$ is projec.

Lemma 4.5.2. *Let $\mathscr{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero. Then $H^n(f)\colon H^n(X) \to H^n(Y)$ is the 0 morphism.*

Proof. Let $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. Then $d_X^n = 0$ on $\ker d_X^n$ and $d_Y^{n-1} \circ s^n = 0$ on $\ker d_Y^n / \operatorname{Im} d_Y^{n-1}$. Hence $H^n(f)\colon \ker d_X^n / \operatorname{Im} d_X^{n-1} \to \ker d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the zero morphism. □

In view of Lemma 4.5.2, the functor $H^0\colon \mathrm{C}(\mathscr{C}) \to \mathscr{C}$ extends as a functor

$$H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}.$$

One shall be aware that the additive category $\mathrm{K}(\mathscr{C})$ is not abelian in general.

qis, def.

Definition 4.5.3. One says that a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic. In particular, $X \in \mathrm{C}(\mathscr{C})$ is qis to 0 if and only if the complex X is exact.

Remark 4.5.4. By Lemma 4.5.2, a complex homotopic to 0 is qis to 0, but the converse is false. In particular, the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 4.5.5. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \oplus_i H^i(X^\bullet)[-i]. \tag{4.5.5}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

Long exact sequence

Lemma 4.5.6. (The "five lemma".) *Consider a commutative diagram:*

$\beta_0(y_0) = f_1(x_1)$

$$\begin{tikzcd}
X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"'] & X^2 \arrow[r, "\alpha_2"] \arrow[d, "f^2"'] & X^3 \arrow[d, "f^3"'] \\
Y^0 \arrow[r, "\beta_0"'] & Y^1 \arrow[r, "\beta_1"'] & Y^2 \arrow[r, "\beta_2"'] & Y^3
\end{tikzcd}$$

$x_2 \in \operatorname{Ker} \alpha_2$,
$\operatorname{Ker} f_2$
$\operatorname{im} \alpha_1 = \operatorname{Ker} \alpha_2$

and assume that the rows are exact.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

According to Convention 4.0.1, we shall assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A. Hence we may choose elements in the objects of $\mathscr{C}$.

$f_2\alpha_1$ sends x_1 to 0. Hence, so does $\beta_1 f_1$.

β_2 sends 0 to 0

mono is left cancellable

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathscr{C}^{\mathrm{op}}$. $\square$

∃ a preimage to an img.

Lemma 4.5.7. (The snake lemma.) *Consider the commutative diagram in $\mathscr{C}$ below with exact rows:*

$$\begin{tikzcd}
 & X' \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[r, "g"] \arrow[d, "\beta"'] & X'' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y'' &
\end{tikzcd}$$

Then there exists a morphism $\delta\colon \ker\gamma \to \operatorname{Coker}\alpha$ giving rise to an exact sequence:

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma. \tag{4.5.6}$$

Proof. here again, we shall assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A.
(i) Let us first prove that the sequence $\ker\alpha \to \ker\beta \to \ker\gamma$ is exact. Let $x \in \ker\beta$ with $g(x) = 0$. Using the fact that the first row is exact, there exists $x' \in X'$ with $f(x') = x$. Then $f' \circ \alpha(x') = \beta \circ f(x') = 0$. Since f' is a monomorphism, $\alpha(x') = 0$ and $x' \in \ker\alpha$.
(ii) The sequence $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$ is exact. If one works in the abstract setting of abelian categories, this follows from (i) by reversing the arrows. Otherwise, if one wishes to remain in the setting of A-modules, one can adapt the proof of (i)[2].
(iii) Let us construct the map δ making the sequence exact. Let $x'' \in \ker\gamma$ and choose $x \in X$ with $g(x) = x''$. Set $y = \beta(\overset{x}{\cancel{y}})$. Since $g'(y) = 0$, there exists $y' \in Y'$ with $f'(y') = y$. One defines $\delta(x'')$ as the image of y' in $\operatorname{Coker}\alpha$, that is, in $Y'/\operatorname{Im}\alpha$.
The reader will check that the map δ is well-defined (i.e., the construction does not depend on the various choices) and that the sequence (4.5.6) is exact. $\square$

One shall be aware that the morphism δ is not unique. Replacing δ with $-\delta$ does not change the conclusion.

Theorem 4.5.8. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathrm{C}(\mathscr{C})$.*

(i) *For each $k \in \mathbb{Z}$, the sequence $H^k(X') \to H^k(X) \to H^k(X'')$ is exact.*

(ii) *For each $k \in \mathbb{Z}$, there exists $\delta^k : H^k(X'') \to H^{k+1}(X')$ making the long sequence*

$$\cdots \to H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \to \cdots \tag{4.5.7}$$

exact. Moreover, one can construct δ^k functorial with respect to short exact sequences of $\mathrm{C}(\mathscr{C})$.

[2] The reader shall be aware that the opposite of an abelian category is still abelian, but the category $\mathrm{Mod}(A)$ is not equivalent to the oposite category $\mathrm{Mod}(A)^{\mathrm{op}}$.

4.7 Derived functors

Let $\mathscr{C}$ be an abelian category satisfying (4.6.11). Recall that $\mathscr{I}_{\mathscr{C}}$ denotes the full additive subcategory of consisting of injective objects in $\mathscr{C}$. We look at the additive category $\mathrm{K}(\mathscr{I}_{\mathscr{C}})$ as a full additive subcategory of the abelian category $\mathrm{K}(\mathscr{C})$.

Theorem 4.7.1. *Assuming (4.6.11), there exists a functor $\lambda\colon \mathscr{C} \to \mathrm{K}(\mathscr{I}_{\mathscr{C}})$ and for each $X \in \mathscr{C}$, a qis $X \to \lambda(X)$, functorially in $X \in \mathscr{C}$.*

Proof. (i) Let $X \in \mathscr{C}$ and let $I_X^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ be an injective resolution of X. The image of $I_X^\bullet$ in $\mathrm{K}^+(\mathscr{C})$ is unique up to unique isomorphism, by Proposition 4.6.6.

Indeed, consider two injective resolutions $I_X^\bullet$ and $J_X^\bullet$ of X. By Proposition 4.6.6 applied to id_X, there exists a morphism $f^\bullet: I_X^\bullet \to J_X^\bullet$ making the diagram (4.6.12) commutative and this morphism is unique up to homotopy, hence is unique in $\mathrm{K}^+(\mathscr{C})$. Similarly, there exists a unique morphism $g^\bullet: J_X^\bullet \to I_X^\bullet$ in $\mathrm{K}^+(\mathscr{C})$. Hence, $f^\bullet$ and $g^\bullet$ are isomorphisms inverse one to each other.
(ii) Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$, let $I_X^\bullet$ and $I_Y^\bullet$ be injective resolutions of X and Y respectively, and let $f^\bullet: I_X^\bullet \to I_Y^\bullet$ be a morphism of complexes such as in Proposition 4.6.6. Then the image of $f^\bullet$ in $\mathrm{Hom}_{\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})}(I_X^\bullet, I_Y^\bullet)$ does not depend on the choice of $f^\bullet$ by Proposition 4.6.6.

In particular, we get that if $g\colon Y \to Z$ is another morphism in $\mathscr{C}$ and $I_Z^\bullet$ is an injective resolutions of Z, then $g^\bullet \circ f^\bullet = (g \circ f)^\bullet$ as morphisms in $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})$. □

Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and recall that $\mathscr{C}$ satisfies (4.6.11). Consider the functors

$$\mathscr{C} \xrightarrow{\lambda} \mathrm{K}^+(\mathscr{I}_{\mathscr{C}}) \xrightarrow{F} \mathrm{K}^+(\mathscr{C}') \xrightarrow{H^n} \mathscr{C}'.$$

extends on the right

Definition 4.7.2. One sets

$$R^nF = H^n \circ F \circ \lambda \tag{4.7.1}$$

and calls R^nF the n-th right derived functor of F.

By its definition, the receipt to construct $R^nF(X)$ is as follows:

- choose an injective resolution $I_X^\bullet$ of X, that is, construct an exact sequence $0 \to X \to I_X^\bullet$ with $I_X^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$,
- apply F to this resolution,
- take the n-th cohomology.

inj res.
$0 \longrightarrow \dots$
proj. res.
$\dots \longrightarrow 0$

In other words, $R^nF(X) \simeq H^n(F(I_X^\bullet))$. Note that

- R^nF is an additive functor from $\mathscr{C}$ to $\mathscr{C}'$,
- $R^nF(X) \simeq 0$ for $n < 0$ since $I_X^j = 0$ for $j < 0$,
- $R^0F(X) \simeq F(X)$ since F being left exact, it commutes with kernels,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if F is exact,
- $R^nF(X) \simeq 0$ for $n \neq 0$ if X is injective, by the construction of $R^nF(X)$.

extends on the left

__Exercise__ Prove that

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{\gcd(m,n)\mathbb{Z}}$$

__Proof__ The definition of $\otimes$ in terms of generators relations is —

$$\left\{\begin{array}{l} (m, n+n') - (m,n) - (m,n') \\ (m+m', n) - (m,n) - (m',n) \\ (am, n) - (m, an) \\ \lambda(m,n) - \lambda(m,n) \end{array}\right.$$

Let $f : \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{m\mathbb{Z}} \otimes \frac{\mathbb{Z}}{n\mathbb{Z}}$

The $\otimes$ is the quotient of the product by the following equiv relations

$$\begin{cases} (m+m')\otimes n = \\ \quad m\otimes n + m'\otimes n \\ m\otimes(n+n') = \\ \quad m\otimes n + m\otimes n' \\ \quad am\otimes n = m\otimes an \\ \lambda m\otimes n = m\otimes \lambda n \\ \quad = \lambda(m\otimes n) \end{cases}$$

It remains to show that these equivalence relations remove the l.c.m from the product, hence yielding the required result —

To take the l.c.m of m and n, where $m \in \frac{\mathbb{Z}}{m\mathbb{Z}}$

and $n \in \frac{\mathbb{Z}}{r\mathbb{Z}}$

Consider the relevant generators

$$\begin{cases} \overline{(am, n) - (m, an)} \\ (\lambda m, n) - \lambda(m, n) \end{cases}$$

Here, $a, \lambda \in \mathbb{Z}$

Replace a by λ —

$$\begin{cases} (\lambda m, n) - (m, \lambda n) \\ (\lambda m, n) - \lambda(m, n) \end{cases}$$

Only $\overline{(\lambda m, n) - (m, \lambda n)}$ is relevant here and it obviously generates the submodule

$$\frac{\mathbb{Z}}{\gcd(m,n)\,\mathbb{Z}}.$$

Exercise

$$\frac{R}{I} \otimes N \xrightarrow{\sim} \frac{N}{IN}.$$

Prove it.

$\frac{R}{I}$ is a right R-module, N is a left R-module.

$$\frac{N}{IN} \qquad IN \in I$$

$$\frac{R}{I} \otimes N \; \underline{\qquad\qquad\qquad}$$

Writing out the generator relations, we get $\underline{\qquad\qquad}$

let $x \in \frac{R}{I}$

and $n \in N$.

$(r, n+n') - (r, n) - (r, n')$

$(r+r', n) - (r, n) - (r', n)$

$(ar, n) - a.(r.n)$

$(r, an) - a.(r, n)$

$a \in R$

To show that these generators generate N/IN

$$
\begin{tikzcd}
\frac{R}{I} \otimes N \arrow[r, "\varphi", "A\text{-linear}"'] & \frac{N}{IN} \\
\frac{R}{I} \times N \arrow[u, "\exists !"'] \arrow[ur, "A\text{-bilinear}"'] &
\end{tikzcd}
$$

$\frac{N}{IN}$ in terms of generators $\longrightarrow$ $n . i = 0$ for $n \in N$, $i \in I$

$\frac{R}{I} \otimes N$ in terms of generators —

$$\left.\begin{array}{l} r \in \frac{R}{I},\ n \in N,\ a \in R \\ (ar, n) - a(r, n) \\ (r, an) - a(r, n) \end{array}\right\}$$

for $i = 0 \cdot I$

$$\varphi : \frac{R}{I} \otimes N \longrightarrow \frac{N}{IN}$$

φ is an A-linear mapping

$\varphi(at) = a\varphi(t)$

$\varphi(t+t') = \varphi(t) + \varphi(t')$

$$\phi : \frac{\overbrace{R \times N}^{P}}{I} \longrightarrow \frac{N}{IN}$$

$\phi(ap) = a\phi(p)$

$\phi(p+p') = \phi(p) + \phi(p')$

Relationship b/w ϕ and φ

$$\begin{tikzcd} \bullet \arrow[r, "\phi"] \arrow[d, dashed, "\exists!\,\theta"'] & \bullet \arrow[dl, bend right, "\xi"'] \\ \bullet \arrow[ur, "\varphi"'] & \end{tikzcd}$$

$$\theta : \frac{R \times N}{I} \longrightarrow \frac{R}{I} \otimes N$$

$\varphi \theta = \phi$.

To show: $\xi = \varphi^{-1}$.

$$\frac{NR}{IN} \otimes N \longrightarrow \frac{N}{IN}$$

| since $\dfrac{R}{I} = \dfrac{RN}{IN}$

$$\underbrace{\frac{NR}{IN}}_{Q} \otimes N$$

Q is an R-module
N is also an R-mod
$a \in R$, $q \in Q$ ———
$n \in N$

$$(aq, n) - a(q, n)$$
$$(q, an) - a(q, n)$$

What is $\frac{N}{IN} \otimes N$

$$\begin{tikzcd}
\frac{\cancel{A}R}{IN} \times N \arrow[r] \arrow[d, "\exists!"'] & \frac{N}{IN} \\
\frac{\cancel{A}R}{IN} \otimes N \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{Z}_5 \otimes \mathbb{Z} = \mathbb{Z}_5$$

$$\frac{\mathbb{Z}}{5\mathbb{Z}} \times \frac{\mathbb{Z}_8}{\mathbb{Z}_8} = \frac{\mathbb{Z} \times \mathbb{Z}_8}{5\mathbb{Z} \times \mathbb{Z}_8}$$

$$\underbrace{\mathbb{Z}_8}_{M} \otimes \underbrace{\mathbb{Z}}_{N} = ?$$

$$(m+m', n) - (m, n) - (m', n)$$

$$(m, n'+n) - (m, n') - (m, n)$$

$$(am, n) - a.(m, n)$$

$$(m, an) - a.(m, n)$$

$a \in \mathbb{Z}$, $m \in \mathbb{Z}_8$, $n \in \mathbb{Z}$

$$(3.m, n) - 3.(m, n)$$

$\gcd(3,4)$

$$\left.\begin{array}{l} a\cdot 3 - a\cdot 12 \\ a\cdot 4 - a\cdot 12 \end{array}\right\}$$

what does this generate?

$\gcd(2,4)$

$(a\cdot 2, 4) - a(2,4)$

$(2, a\cdot 4) - a(2,4)$

Let J be an ideal of N.

I is an ideal of R

I is not necessarily an ideal of N.

$$JN \subseteq J$$

$$IR \subseteq R$$

$$\frac{R\cancel{N}}{IN} \otimes N \qquad RN = R$$

$$= \frac{R}{IN} \otimes N = \frac{N \otimes R}{IN}$$

$$\frac{\mathbb{Z}}{8\mathbb{Z}} \otimes \mathbb{Z} = \frac{\mathbb{Z} \times \mathbb{Z}}{8\mathbb{Z}}$$

$$= \frac{\mathbb{Z}}{8\mathbb{Z}} = \mathbb{Z}_8$$

$$\frac{N \otimes R}{IN} = \frac{N}{IN} \quad \blacksquare$$

<u>Ex</u> Given two additive functors

$$\left.\begin{array}{l} F : A \longrightarrow B \\ U : A \longrightarrow B \end{array}\right\} \text{b/w abelian cat}$$

exact

Prove that $U(L_i F) \cong L_i(UF)$ and $U(R^i F) \cong R^i(UF)$

<u>Proof</u> U is exact and commutes with ker. L_i and R^i are $H_n^0 \ldots$ and $H^n{}_0 \ldots$, which are again some ker/some im.

<u>Todo</u>: Def. of cohomology.

Ex Show that
$L_m L_n \mathcal{F} = 0$.

Proof: $L_i F = \mathcal{H}_n \circ F \circ \lambda$
where $\lambda = C \longrightarrow K(\ldots)$

$L_m L_n \mathcal{F}$
$= \mathcal{H}_m \circ \mathcal{H}_n \circ \mathcal{F} \circ \lambda$.
$= 0$.

Ex Construct an
injective resolution

of $\mathbb{Z}_p$ with respect to the functor $\mathbb{Z}_p \otimes -$

$\mathcal{F} : \mathbb{Z}_p \otimes -$, $\mathcal{C} : \mathbb{Z}$

$\mathbb{Z}_p \otimes P$ should be exact.

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_p \longrightarrow 0$$

$$\begin{tikzcd}
\mathbb{Z}_p \times \cdot \arrow[d] \arrow[dr] & \\
\mathbb{Z}_p \otimes \cdot \arrow[r] & I
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z}_p \times \cdot \arrow[d] \arrow[dr] & \\
\mathbb{Z}_p \otimes \cdot \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}_p \to 0$$

<u>Ex</u> Show that, for p prime, there are p extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$.

a) The split extension —

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p \oplus \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p$$

b) The $p-1$ equiv. classes —

$$\mathbb{Z}_p \xhookrightarrow{\times \frac{1}{i\mathbb{Z}}} \mathbb{Z}_{p^2} \xrightarrow{\times i\mathbb{Z}} \mathbb{Z}_p$$

$$\frac{\mathbb{Z}}{p^2\mathbb{Z}} \times i\mathbb{Z} = \frac{\mathbb{Z}}{\left(\frac{p^2}{i}\right)\mathbb{Z}} \qquad i = 1, 2, \ldots, p-1$$

Ex ξ is an extn.

$\theta(\xi)$ is used to denote the image of 1_A in the morphism:

$$\mathrm{Hom}(A,A) \xrightarrow{f} \mathrm{Ext}^1(A,B)$$

$$\mathrm{Ext}^1(A,B) =$$

$$\left(R^1\mathrm{Hom}(A,-)\right)(B) = 0$$

$$0 \longrightarrow A \longrightarrow A$$

$$0 \longrightarrow \mathrm{Hom}(A,-) \longrightarrow \mathrm{Hom}(A,-)$$

$$0 \longrightarrow A \hookrightarrow B \longrightarrow A \longrightarrow B$$

$\ker(\operatorname{Hom}(A,B)) = \operatorname{Ext}^1(A,B)$
$= 0$

$\Rightarrow A \subseteq B$.

$\theta(\xi) = 0 \Rightarrow \theta$ is a homology/cohomology function operating on ξ. Obviously, $\theta(\xi) = 0$
$\Leftrightarrow \xi$ is scindée.

There is but one possible choice for ξ —

$$B \longrightarrow A \oplus B \longrightarrow A$$

is the only possible choice for ξ.

$Ext^1(A, B) = \ker(Hom(A, B)) = 0.$

<u>Ex</u> From the above $\theta(\xi) = \ker(Hom(A, B))$ and $\xi : B \longrightarrow A \oplus B \longrightarrow A$ deduce that ξ and ξ' are equiv. $\Rightarrow \theta(\xi) = \theta(\xi')$

$$\begin{tikzcd}
\xi : B \arrow[r, hook, "p"] \arrow[d, "1_B"] & P \arrow[r, two heads, "q"] \arrow[d, "f"] & A \arrow[d, "1_A"] \\
\xi' : B \arrow[r, hook, "r"] & Q \arrow[r, two heads, "s"] & A
\end{tikzcd}$$

ξ and ξ' are equiv.
$\Rightarrow$ the above dia. is commutative.

$q = sf$ and $r = fp$

$\theta(\xi) = \mathrm{Ker}(\mathrm{Hom}(A, B))$

$\theta(\xi') = \mathrm{Ker}(\mathrm{Hom}(A, B))$

Hence $\theta(\xi) = \theta(\xi')$
since ξ is equiv. to ξ'.

Ex Show that

θ induces a bijection b/w $Ext^i(A,B)$ and $Ext^1(A,B)$ given

$\theta(\xi) = Ker(Hom(A,B))$

and ξ is an extn.

Proof $\theta(\xi)$ is the image of $Hom(A,A)$ in $Ext^n(A,B) =$

$Ker(Hom(A,B))$

$Ext^i(A,B) = R^i(Hom(A,-))(B)$

$= H^n P(\operatorname{Hom}(A, -))(B)$

$0 \longrightarrow A \hookrightarrow B \xrightarrow{0} A \hookrightarrow B$

$B \supseteq A$.

Every H^i in $\operatorname{Ext}^i(A, B)$
is either A or 0.

$H^1 P(\operatorname{Hom}(A, -))(B)$
is $\ker(\operatorname{Hom}(A, B)) = 0(\xi)$

0 is exactly H^i.

The $\ker(\operatorname{Hom}(A, B))$
is either A or 0 in
$\operatorname{Ext}(A, B)$.

There are two cases
— $\ker(\operatorname{Hom}(A,B)) = 0$
when $B \not\supseteq A$ and
$\ker(\operatorname{Hom}(A,B)) = A$
when $B \subseteq A$.

Ex Find the number of classes of equiv. of the extn. of $\mathbb{Z}_4$ by $\mathbb{Z}_2$ in the $\mathbb{Z}_8$ module

a) The split extn.

$$\mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 .$$

b) $\mathbb{Z}_2 \xrightarrow{\times i\mathbb{Z}} \mathbb{Z}_8 \xrightarrow{\times i\mathbb{Z}} \mathbb{Z}_4$

$i = 1, 2, 3.$

Hence, there are four.

<u>Ex</u> $R = \mathbb{Z}_m$, $A = \mathbb{Z}_d$, $d \mid m$, $m \geq 2$. Find $\mathrm{Ext}^i(A, B) \ncong B$.

$$\mathrm{Ext}_R^i(A, B) = \left(R^i \operatorname{Hom}(A, -)\right)(B)$$

$$0 \to A \xrightarrow{q} B \xrightarrow{0} A \xrightarrow{r} B$$

$A = \mathbb{Z}_d$

$B = ?$

$R = \mathbb{Z}_m$

$d \mid m$. $m \geq 2$.

$\mathrm{Ext}^0(A, B) = {}_dB$

$\mathrm{Ext}^i(A, B) = \left(\frac{m}{d} B\right) \Big/ (d.B)$

— odd

$\mathrm{Ext}^i(A, B) = \left({}_dB\right) \Big/ \left(\frac{m}{d} \cdot B\right)$

— even — To show

$$0 \to \mathbb{Z}_d \xrightarrow{d_0} B \xrightarrow{d_1} \mathbb{Z}_d \xrightarrow{d_2} B$$

odd case: $\dfrac{\ker d_1}{\operatorname{im} d_0}$

0 case: $\ker d_0 = {}_dB$

even case: $\dfrac{\ker d_2}{\operatorname{im} d_1}$

$\operatorname{im} d_0 = d \cdot B$	$\operatorname{im} d_2 = \frac{m}{d} \cdot B$
$\ker d_0 = {}_dB$	$\ker d_2 = {}_dB.$
$\operatorname{im} d_1 = d \cdot B$	
$\ker d_1 = {}_{\frac{m}{d}}B$	

The kernel of $\mathbb{Z}_d \longrightarrow \mathbb{Z}$ is $d\mathbb{Z}$. Hence, the kernel of $\mathbb{Z}_d \longrightarrow B$ is $_dB$.

The image of $\mathbb{Z} \longrightarrow \mathbb{Z}_d$ is $d \cdot \mathbb{Z}$

Hence, the image of $B \longrightarrow \mathbb{Z}_d$ is $d \cdot B$.

The kernel of the

map $\mathbb{Z} \longrightarrow \mathbb{Z}_d$ is $\frac{1}{d}\mathbb{Z}$. Hence the kernel of $B \longrightarrow \mathbb{Z}_d$ is $\frac{m}{d} B$, since B is a $\mathbb{Z}_m$-module. The rest follows.

<u>Ex</u> $A_G = \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A^G = \mathrm{Hom}_G(\mathbb{Z}, A)$ Prove it.

Any Ab can be considered as a trivial G-mod as follows:

$$\mathrm{Ab} \longrightarrow G\text{-Mod}$$

$$\{ga \mid ga = a\}$$

There exist two functors from

G-mod $\longrightarrow$ Ab

forgetful

(a) Invariant subgroup

$$\{a \mid ga = a\} = A^G$$

(b) Covariant subgroup —

$A_G = A /$ submod generated by $\{(ga - a), g \in G, a \in A\}$

Any group G can be considered a trivial G-mod.

G-mod can be interpreted as a $\mathbb{Z}G$-module (integral group ring) or Ab^G (the functor category)

$\mathbb{Z}$ can be considered as either a $\mathbb{Z}$-module or a $\mathbb{Z}G$-module.

Let us interpret $\mathbb{Z}$ as a $\mathbb{Z}-\mathbb{Z}G$ bimodule.

$$\begin{tikzcd}
(-)_G \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[drr, shift left] \arrow[drr, phantom, "\perp"] & (-)^G \arrow[l, shift left] & \\
 & & \mathrm{Ab} \arrow[r] \arrow[ull, shift left] & G\text{-Mod}
\end{tikzcd}$$

the coinvariant group is left adjoint to the functor that sends Ab to its trivial G-module (the trivial G-Mod functor).

Now, consider $\mathbb{Z}$ as a $\mathbb{Z}-\mathbb{Z}G$ bimodule — The functor from

$$\mathbb{Z}\text{-mod} \longrightarrow \mathbb{Z}G\text{-mod}$$

is $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$. The left adjoint to this is $\mathbb{Z} \otimes_{\mathbb{Z}G} (-)$.

The functor from $\mathbb{Z}$-mod to $\mathbb{Z}G$-mod is the trivial G-mod functor. It is right adjoint to $(-)_G$.

$$\begin{tikzcd}
(-)_G \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[d, equal, "\checkmark"] & \mathbb{Z}\text{-mod} \to \mathbb{Z}G\text{-mod} \arrow[l, shift left] \\
\mathbb{Z} \otimes_{\mathbb{Z}G} (-) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbb{Z}\text{-mod} \to \mathbb{Z}G\text{-mod} \arrow[l, shift left]
\end{tikzcd}$$

— .

It remains to show that $A^G = \mathrm{Hom}_G(\mathbb{Z}, A)$

$$A^G = \mathrm{Hom}_{Ab}(\mathbb{Z}, A^G)$$

$$= \mathrm{Hom}_G(\mathbb{Z}, A).$$

<u>Ex</u> Show that $m H^n(G; M) = 0$ for G a group of order m, and M a G-module.

$$m H^n(G; M) = m(\mathcal{L}^i \mathrm{Hom}(G, -))(M)$$

$$\mathcal{G} \xrightarrow{\times \frac{1}{m}} \mathcal{G} \xrightarrow{\times m} \mathcal{G} \longrightarrow 0$$

$$\frac{\mathcal{G}}{m\mathcal{G}} = H_1 \qquad H_n = \frac{{}_m\mathcal{G}}{m\mathcal{G}} \quad \forall n \geq 2$$

$$mH_1 = 0$$

$$mH_n = \frac{m\mathcal{G}}{\frac{\mathcal{G}}{m}} = 0.$$

$$\mathcal{G} \xrightarrow{\varphi} M \xrightarrow{\psi} \mathcal{G} \xrightarrow{\varphi} M \longrightarrow 0$$

$$\left.\begin{array}{l} \varphi : \mathcal{G} \longrightarrow M \\ \psi : M \longrightarrow \mathcal{G} \end{array}\right\} \begin{array}{c} \longrightarrow \\ \perp \\ \longleftarrow \end{array}$$

$$H_2(G;M) = \frac{\ker \varphi}{\operatorname{im} \psi}$$

$$= \frac{\ker \{gm,\ gm = m\}}{\operatorname{im} M / (gm - m)_{\text{submod}}}$$

$$= \frac{\{gm - m = 0\}}{\{m \mid gm - m = 0\}}$$

$$m \cdot H_2(G;M) = 0$$

$$m \cdot H_n(G;M) = 0 \ \forall \ n \geq 2$$

Ex For $G = \frac{\mathbb{Z}}{n\mathbb{Z}} = \{\omega^i;\ i = 0, \ldots n-1\}$

$$N = \omega - 1, \quad T = \sum_{0}^{n-1} \omega^i$$

Show that

$$\mathbb{Z}G \xrightarrow{\times T} \cdots \xrightarrow{\times N} \mathbb{Z}G \to 0$$

is a free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module.

By Laurent expansion

$$\frac{1}{\omega - 1} = \frac{1}{\omega} \sum_{0}^{\infty} \left(\frac{1}{\omega}\right)^n$$

or $$\frac{1}{1-\omega} = \sum_{0}^{\infty} \omega^n$$

Here, we have a group of order n, so that the series can stop at $n-1$.

$$\frac{1}{1-\omega} = \sum_{0}^{n-1} \omega^i = N$$

The augmentation

ideal J corresponding to a free module resolution is $\{w - 1\}$.

Hence, $T = w - 1$ — J

$$N = \sum_{0}^{n-1} w^i$$

The Laurent expansion of $\frac{1}{1-w}$

$\left(\frac{1}{1-w}\right)$ and $w - 1$ together generate G.

$$\sum \lambda_i (\omega - 1) + \frac{1}{\varepsilon_i}\frac{1}{1-\omega}$$

$$= \sum \omega^n$$

$$\sum \frac{\lambda_i(\omega-1)}{\varepsilon_i(1-\omega)} = \sum \omega^i$$

$$\sum \frac{(1+\omega)(\omega-1)\lambda_i}{(1-\omega^2)\varepsilon_i}$$

$$= \sum \omega_i$$

ω_i is in the ring $\mathbb{Z}/n\mathbb{Z}$

$$\Rightarrow \sum \frac{-\lambda_i}{\varepsilon_i} = \sum \omega_i$$

$$\sum -\frac{x_i}{\varepsilon_i} = \sum \omega^i$$

where ω is a $\mathbb{N}$. Hence, we have shown that

$\left(\frac{1}{1-\omega}\right)$ and $(\omega - 1)$

together form a basis for $\overline{\sum \omega_i} = G$

The free resolution follows.

Ex Find $H^n(\mathbb{Z}_n, \mathbb{Z})$ and $H_n(\mathbb{Z}_n, \mathbb{Z})$

$\operatorname{Tor}_i(\mathbb{Z}_n, \mathbb{Z})$
$= H_i L_i(\mathbb{Z}_n \otimes -)(\mathbb{Z})$

$\operatorname{Ext}_i(\mathbb{Z}_n, \mathbb{Z})$
$= H_i R_i(\operatorname{Hom}(\mathbb{Z}_n, -)),$
$(\mathbb{Z})$

$0 \longrightarrow \mathbb{Z}_n \xrightarrow{\times \frac{1}{n}} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}_n$

$H_1 = \frac{1}{n}\mathbb{Z}_n = 0.$

$$H_2 = \frac{n\mathcal{Z}}{\frac{\mathcal{Z}_n}{n}} = \frac{n\,n\mathcal{Z}}{\mathcal{Z}_n}$$

$$H_3 = \frac{\frac{1}{n}\mathcal{Z}\,n}{} = 0.$$

$$\cdots$$

$$H_{even} = \frac{n\,n\mathcal{Z}}{\mathcal{Z}_n}$$

$$H_{odd} = 0$$

$$H_1 = 0.$$

To find — $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z})$

$$= (H_i L_i(\mathbb{Z}_n \otimes -))(\mathbb{Z})$$

$$\begin{tikzcd}
\mathbb{Z}_n \times \mathbb{Z} \arrow[d] \arrow[dr] & \\
\mathbb{Z}_n \otimes \mathbb{Z} \arrow[r] & \mathbb{Q}
\end{tikzcd}$$

$\cancel{\mathbb{Z}_n \otimes \mathbb{Z}} \longrightarrow \cancel{\mathbb{Z}_n \times \mathbb{Z}} \longrightarrow$

$\cancel{\mathbb{Z}_n \otimes \mathbb{Z}} \longrightarrow 0$

$$\mathbb{Z}_n \xrightarrow{\times \frac{1}{n}} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}_n \to 0$$

$$H_1 = \frac{\mathbb{Z}_n}{n\mathbb{Z}}$$

$$H_2 = \frac{n^{\mathbb{Z}} \cdot n}{\mathbb{Z}_n}$$

$$H_3 = \frac{1/n^{\mathbb{Z}_n}}{n\mathbb{Z}} = 0$$

$$H_{odd} = 0$$

$$H_{even} = \frac{n^{\mathbb{Z}} \cdot n}{\mathbb{Z}_n}$$

$$H_1 = \mathbb{Z}_n / n\mathbb{Z}$$

Ex Given the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p"] & \mathbb{Z}_{p^2} \arrow[r] & \mathbb{Z}_p \arrow[d] \\ & & & 0 \end{tikzcd}$$

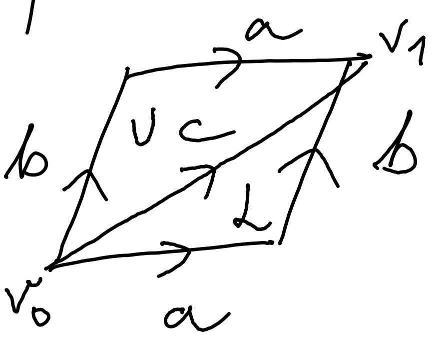
induces a cohomology (where X is a topological space): [singular coh.]

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X, \mathbb{Z}_p) \arrow[r] & H^n(X, \mathbb{Z}_{p^2}) \arrow[d] \\ \cdots & H^{n+1}(X, \mathbb{Z}_p) \arrow[l] & H^n(X, \mathbb{Z}_p) \arrow[l, "\beta"'] \end{tikzcd}$$

Find the coboedism operator β.

<u>Proof</u>: ______________________.

<u>Ex</u> Define the simplicial homology for the torus.



$\Delta_0 : \{v_0, v_1\}$

$\Delta_1 : 2\{c\} = 2\{a+b\}$

$\Delta_2 : \{v, \alpha\}$

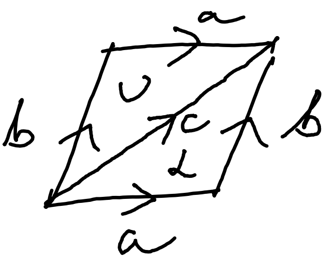
$\operatorname{im} \partial_0 = 0$ since it is an α^i

$\partial_1 : \Delta^1 \longrightarrow \Delta^0$

$2\{c\} = 2\{a+b\} \xrightarrow{\partial_1} \{v_0, v_1\}$

$\operatorname{im} \partial_1 = 0$

$\ker \partial_1 = \mathbb{Z}$



$\partial_2 : \Delta^2 \longrightarrow \Delta^1$

$\{U, L\} \xrightarrow{\partial_2} 2\{c\}$

$= 2\{a+b\}$

$\{+c - a - b, -c - a - b\}$

$\xrightarrow{\partial_2} 2\{c\} = 2\{a+b\}$

$\{c - a - b, -c - a - b\}$

$\xrightarrow{\partial_2} 2\{c\} = 2\{a+b\}$

$c = a + b$

$\{c - a - b, -c - a - b\}$

$\xrightarrow{\partial_2} 2\{c\} = 2\{a+b\}$

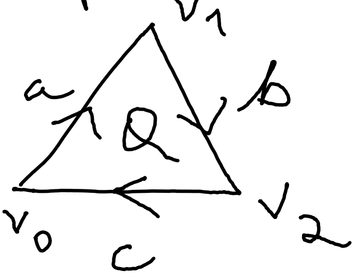
$\operatorname{im} \partial_2 = 0$

$\operatorname{Ker} \partial_2 = \mathbb{Z} \oplus \mathbb{Z}$

$$H_1 = \frac{\operatorname{Ker} \partial_1}{\operatorname{im} \partial_0} = \frac{\mathbb{Z}}{0} = \mathbb{Z}$$

$$H_2 = \frac{\operatorname{Ker} \partial_2}{\operatorname{im} \partial_1} = \mathbb{Z} \oplus \mathbb{Z}$$

<u>Ex</u> Find the homology of a circle — S^1.



$a + b = c$

$\Delta^0 : \{v_0, v_1, v_2\}$

$= \{v_0\}$

$\Delta^1 : \{a, b, c\}$, $\Delta^2 : Q$

$\operatorname{im} \partial_0 = \mathbb{Z}$

$\partial_1 : \Delta^1 \longrightarrow \Delta^0$

$\{a, b, c\} \xrightarrow{\partial_1} \{v_0, v_1, v_2\}$

$2\{c\} = 2\{a+b\} \xrightarrow{\partial_1} \ldots$

$\operatorname{Ker} \partial_1 = \mathbb{Z}$

$\operatorname{im} \partial_1 = 0$

$\partial_2 : \Delta^2 \longrightarrow \Delta^1$

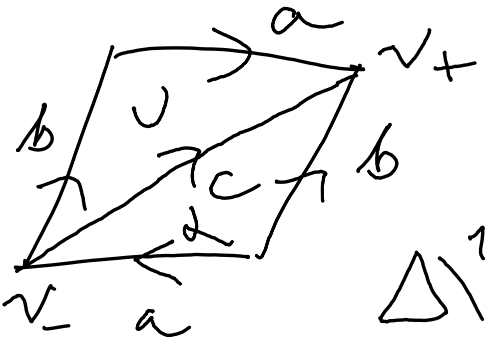
$\{-c-a-b\} \longrightarrow 2\{c\}$

$= 2\{a+b\}$

$\operatorname{Ker} \partial_2 = \mathbb{Z}$

$\operatorname{im} \partial_2 = 0$

Ex Derive the homology groups of a Klien bottle.



$a+b \neq c$

$\Delta^0 : \{v_+, v_-\}$

$\Delta^1 : \{c, a, b\}$

$\Delta^2 : \{U, L\} = \{-c-a-b, -c-a+b\}$

$\operatorname{im} \partial_0 = 0$

$\partial_1 : \Delta^1 \longrightarrow \Delta^0 \quad \{a, -a\} \longrightarrow \{v_+, v_-\}$

$\{a, b, c\} \longrightarrow \{v_+, v_-\}$

$\operatorname{Ker} \partial_1 = \mathbb{Z} \oplus \mathbb{Z}_2$

$\operatorname{im} \partial_1 = \mathbb{Z}$

$\partial_2 : \Delta^2 \longrightarrow \Delta^1$

$\{-c-a-b, -c-a+b\}$

$\xrightarrow{\partial_2} \{a, b, c\}$

$\{-c-a-b, -c-a+b\}$

$\longrightarrow \{c+a, -c-a-b,$

$-c-a+b\}$

$\operatorname{Ker} \partial_2 = 0$

$\operatorname{im} \partial_2 = 2\mathbb{Z}$

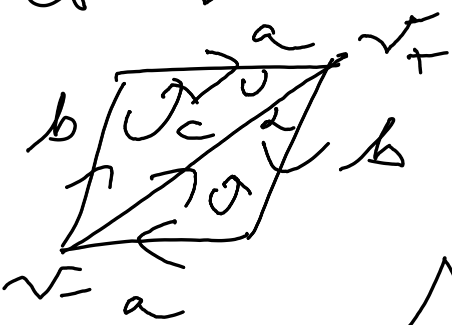
$H_1 = \dfrac{\operatorname{Ker} \partial_1}{\operatorname{im} \partial_0} = \mathbb{Z} \oplus \mathbb{Z}_2$

$$H_2 = \frac{\ker \partial_2}{\operatorname{im} \partial_1} = 0.$$

$$H_3 = \frac{\ker \partial_3}{\operatorname{im} \partial_2} = 0.$$

$$H_0 = \mathbb{Z}.$$

<u>Ex</u> Define simplicial chomology for $\mathbb{RP}^2$.



$\Delta^0 : \{v_+, v_-\}$

$\Delta^1 : \{a, b, c\}$

$= \{c, -c\} = \{a+b, -a-b\}$

$\Delta^2 : \{v, \lambda\}$

$v : \{c - a - b\}$

$\lambda : \{-c - a - b\}$

$\operatorname{im} \partial_0 = 0$

$H_0 = \mathbb{Z}$

$\partial_1 : \Delta^1 \longrightarrow \Delta^0$

$\{c, -c\} = \{a + b, -a - b\}$

$\longrightarrow \{v_0, v_1\}$

There are two maps from $\{c, -c\} \longrightarrow \{v_0, v_1\}$

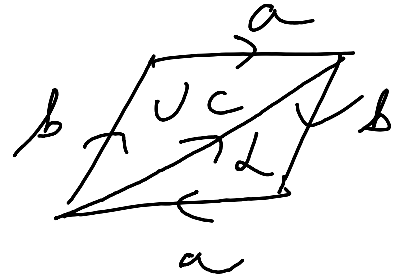
The first is $\{v_0, -c\} \longrightarrow \{v_0, v_1\}$

and the second is $\{c, v_1\} \to \{v_0, v_1\}$

The image of the two maps together

$\operatorname{im} \partial_1 = \mathbb{Z}$

$\operatorname{Ker} \partial_1 = \mathbb{Z}_2$



$\partial_2 : \Delta^2 \longrightarrow \Delta^1$

$\{+c-a-b, -c-a-b\}$

$\xrightarrow{\partial_2} \{2c, -2c\}$

$\{c-a-b, -c-a-b\}$

$\xrightarrow{\partial_2} \{a+b, -a-b\}$

$\operatorname{im} \partial_2 = \mathbb{Z} \oplus \mathbb{Z}$

$\operatorname{Ker} \partial_2 = 0$

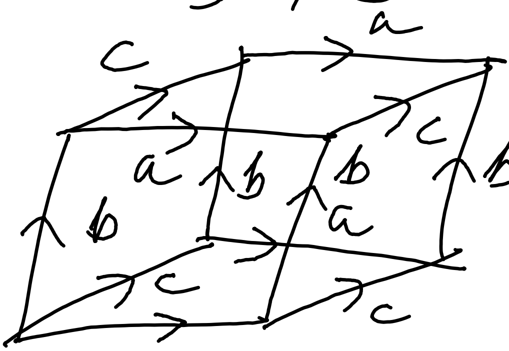
$H_0 = \mathbb{Z}$

$H_1 = \mathbb{Z}_2$

$H_2 = 0$

$H_n = 0 \quad \forall n \geq 3$

<u>Ex</u> Derive the cellular homology groups for $T^3 = S^1 \times S^1 \times S^1$



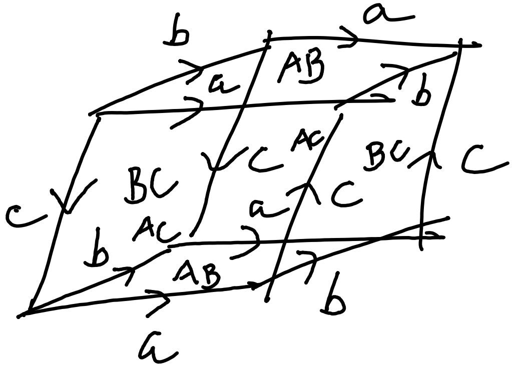
$\left\{\begin{array}{l} 1 \text{ 3-cell} \\ 3 \text{ 2-cells} \\ 3 \text{ 1-cells} \\ 1 \text{ 0-cell} \end{array}\right.$

The cellular chain is

$$\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

$d_1 = d_2 = d_3 = 0$ by local degree arguments. H_i hence coincides with the cellular chain. $H_1 = H_4 = \mathbb{Z}$ and $H_2 = H_3 = \mathbb{Z}_3$.

Ex Derive cellular homology for $K \times S^1$



$\begin{cases} 1 & 0\text{-cell} \\ 3 & 1\text{-cells} \\ 3 & 2\text{-cells} \\ 1 & 3\text{-cell} \end{cases}$

$$\mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \to 0$$

is the cellular map.

$d_1 = 0$ since there is only one zero cell and the attaching map $S^0 \longrightarrow S^0$ corresponding to $\{a, b, c\}$ attached to this vertex is 0.

$d_2 : \{AB, BC, AC\}$
$\to \{a, b, c\}$
$d_2(e^n_\beta) = \sum d_{\alpha\beta} e^n_\alpha$
AB and AC attach to
$\{a, b, c\}$ via wedges
— they are homeomorphic locally, and
the local degree at
each opposite face
has opposite signs,

cancelling out.

$$d_2(\{BC\}) = 2\{C\}$$

as the local degrees add up, yielding a degree of 2 for this map.

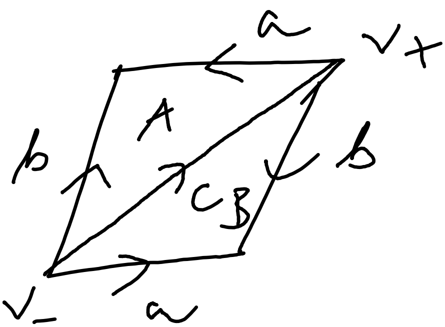
$$d_3 : \{ABC\} \to \{\cancel{AB}, BC, \cancel{AC}\}$$

$$d_3(\{ABC\}) = 2\{BC\}$$

as the local degrees add up.

<u>Ex</u> Calculate

$H^n(\mathbb{RP}^2, \mathbb{Z}_2)$



Il faire cohomologie singulaire (simplicial)

$\Delta^0 = \{v_-, v_+\}$ $= \{a+b, -a-b\}$

$\Delta^1 = \{a, b, c\}$ $\{c, -c\} \longrightarrow$

$\Delta^2 = \{A, B\}$ $\{-v_-, v_+\}$

$d_1 : \{a, b, c\} \xrightarrow{\cancel{\alpha}} \{v_-, v_+\}$

$d_2 : \{A, B\} \xrightarrow{d_2} \{a, b, c\}$

$\{c-a-b, -c+a-b\} \longrightarrow \{?\}$

$$\left\{-b+(a-c),\ -b-(a-c)\right\} \longrightarrow \{a,b,c\}$$

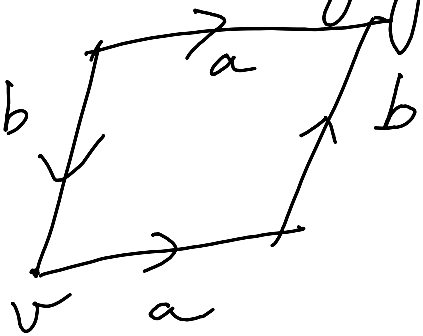
$$\left\{-b+(a-c),\ -b-(a-c)\right\} \xrightarrow{d_2} \left\{+2b,\ -b-(a-c),\ -b+(a-c)\right\}$$

$$\left.\begin{array}{l} \operatorname{Ker} d_2 = 0 \\ \operatorname{im} d_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}\right\}$$

$$\left.\begin{array}{l} \operatorname{Ker} d_1 = \mathbb{Z}_4 \\ \operatorname{im} d_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{array}\right\}$$

$$H_1 = \mathbb{Z}_4,\ H_2 = 0$$

<u>Ex</u> Compute cellular homology of K.



1 0 - cell

2 1 - cells

1 2 - cell

$$\mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

$$d_n(e_\beta^{n+1}) = \sum d_{\alpha\beta}(e_\alpha^n)$$

$$d_n(e_\beta^{n+1}) = \sum d_{\alpha\beta}(e_\alpha^n)$$

degree

$d_1(a) = 0$, since both local degrees cancel.
$d_1(b) = 2v$, as the local degrees add up
$d_2(K) = 2b$, as the terms with 'a' cancel, and the terms with 'b' add up — the local degrees can be said to add up in a wedge sum.

$\operatorname{Ker}(d_1) = \mathbb{Z} \oplus \mathbb{Z}_2$
$\operatorname{im}(d_1) = 2v$ (or $2\mathbb{Z}$)
$\operatorname{Ker}(d_2) = 0$
$\operatorname{im}(d_2) = 2b$ (or $2\mathbb{Z}$)

$H_2 = 0$, $H_1 = \mathbb{Z} \oplus \mathbb{Z}_2$

$H_0 = \mathbb{Z}$.

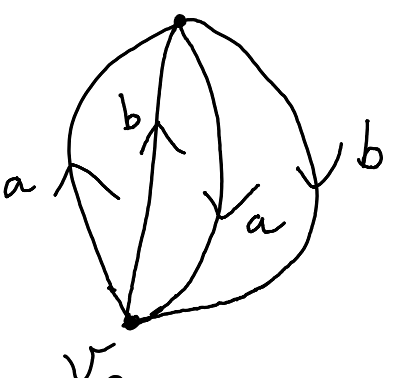
<u>Ex</u> Topological space X is obtained from $S^2 \times S^2$ by attaching a 3-cell to $\{x_0\} \times S^2$ in degree 2. Compute cellular homology of X.

$S^2 \times S^2 \longrightarrow \begin{cases} 1 \text{ 4-cell} \\ 2 \text{ 2-cells} \\ 1 \text{ 0-cell} \end{cases}$

attach to $\{x_0\} \times S^2$ a 3-cell $\longrightarrow$ (+ 1 3-cell)

$$\mathbb{Z} \xrightarrow[0]{d_4} \mathbb{Z} \xrightarrow[\times 2]{d_3} \mathbb{Z}^2 \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \to 0$$

is the cellular map



$H_0 = \mathbb{Z}$

$H_1 = 0$

$H_2 = \mathbb{Z} \oplus \mathbb{Z}_2$

$H_3 = 0 \quad H_4 = \mathbb{Z}$

✓

Ex Compute cohomology of above question

$$0 \xrightarrow[d_1]{0} \mathbb{Z} \xrightarrow[d_2]{0} 0 \xrightarrow[d_3]{0} \mathbb{Z}^2 \xrightarrow[d_4]{\binom{0}{2}} \mathbb{Z} \xrightarrow[d_5]{0} \mathbb{Z}$$

$H^0 = \mathbb{Z}$

$H^1 = 0$

$H^2 = \mathbb{Z}$

$H^3 = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$

$H^4 = \mathbb{Z}$

Ex Compute the cohomology of $S^2 \vee S^2 \vee S^4$ with a 3-cell attached to the second S^2 deg. 2.

$S^2 \vee S^2 \vee S^4$ + 1 3-cell

1 0-cell
0 1-cells
2 2-cells
1 3-cell
1 4-cell

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 \\ 2 \end{pmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$H^0 = \mathbb{Z}$
$H^1 = 0$
$H^2 = \mathbb{Z}$
$H^3 = \mathbb{Z}/2\mathbb{Z}$
$H^4 = \mathbb{Z}$

<u>Ex</u> Compute the cohomology of the above problem in coeff. $\mathbb{Z}_2$

$$0 \to \mathbb{Z}_2 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2$$

$H^0 = \mathbb{Z}_2$

$H^1 = 0$

$H^2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$H^3 = \mathbb{Z}_2$

$H^4 = \mathbb{Z}_2$

<u>Ex</u> Let R be a commutative ring and let $I = (r)$ be an ideal for r not a zero divisor. Show that I is a free R-module.

$I . r \neq 0$, $Ir \subseteq I$

To show: $I \cong R^n$ for some n

$I = rR = (r)$.

$$\cong \underbrace{R \times R \times R \times \dots \times R}_{r \text{ times}}$$

$$\cong R^r$$

Ex Show that $\mathrm{Tor}^1_R(R/I, M) = {}_rM$ and $\mathrm{Ext}^1_R(R/I, M) = M/IM$ for any R-module M.

$$\mathrm{Tor}^1_R(R/I, M) = \frac{R}{I} \otimes M$$

$$= R_r \otimes M$$

$$= {}_rM.$$

$$\mathrm{Ext}^1_R(R/I, M) =$$

$$\mathrm{Hom}\left(\frac{R}{I}, M\right) = \mathrm{Hom}(R_r, M)$$

$$= \frac{M}{(rR)M} = \frac{M}{IM}.$$

<u>Ex</u> Show that the number of equiv. classes of $\mathbb{Z}_m$ by $\mathbb{Z}_n$ is $\gcd(m, n)$

$$\begin{array}{ccc} \mathbb{Z}_m & \xrightarrow[f]{i \text{ maps}} & \mathbb{Z}_m \oplus \mathbb{Z}_n \\ & & g \downarrow i \text{ maps} \\ & & \mathbb{Z}_n \end{array}$$

$f : \mathbb{Z}_m \hookrightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n$

There are $\gcd(m,n)$ such maps.

$g : \mathbb{Z}_m \oplus \mathbb{Z}_n \twoheadrightarrow \mathbb{Z}_n$

There are again $\gcd(m,n)$ such maps.

<u>Ex</u> A ring R is heriditary if all submodules of proj. R-modules are projective. Show that a PID is heriditary.

Let M be a PID, so that every ideal (or submodule) is generated by exactly one element of M: every submodule of M has a single basis. Let J be an ideal (submodule) generated by (j). Let N be another PID.

$$\begin{tikzcd}
 & P \arrow[r, "\psi"] \arrow[d, "\alpha"] \arrow[dl, dashed, "r"'] & & Q \arrow[d, "\beta"] \arrow[dl, dashed, "s"'] \\
N \arrow[r, two heads, "p"] & J \arrow[r] \arrow[rr, bend right, "i"'] & N \arrow[r, two heads, "q"] & M
\end{tikzcd}$$

Since $N \twoheadrightarrow J \rightarrow N$ has to be id_N, and therefore since

$J \to N$ is injective, the image of J in N has basis (λ_j). Since $P \xrightarrow{\alpha} J$ is a well-defined homomorphism with basis (λ_j), $P \xrightarrow{\alpha} J$ must be surjective.

$$\begin{tikzcd}
 & P \arrow[rr, "\psi"] \arrow[dl, dashed, "r"'] \arrow[d, two heads, "\alpha"] & & Q \arrow[dl, dashed, "\exists ! s"'] \arrow[d, "\beta"] \\
N \arrow[r, two heads, "p"] & J \arrow[r, hook, "\varepsilon"] \arrow[rr, bend right, "i"'] & N \arrow[r, two heads, "q"] & M
\end{tikzcd}$$

$q \varepsilon \alpha$ is surjective $= \beta \psi$.

Note that this is our choice of ψ.

$\Rightarrow \beta$ must be surjective as well.

$$\begin{tikzcd}
 & P \arrow[r, "\psi"] \arrow[d, two heads, "\alpha"] \arrow[dl, dashed, "r"'] \arrow[dr, phantom, "\circlearrowleft"] & Q \arrow[d, two heads, "\beta"] \\
N \arrow[r, two heads, "p"] & J \arrow[r, hook, "i"] & M
\end{tikzcd}$$

pr is surjective

$$\{\lambda_1, \ldots, \lambda_n\} \xrightarrow{p} \!\!\!\to \{\lambda_j\}$$

$$\xhookrightarrow{i} \{\lambda_1, \ldots \lambda_m\}$$

$$\begin{tikzcd} & Q \arrow[dl, dashed, "\exists ! s"'] \arrow[d, two heads, "\beta"] \\ N \arrow[r, two heads, "q"] & M \end{tikzcd}$$

$qs = \beta$.

To show:

$pr = \alpha$.

By commutativity of

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, two heads, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & Q \arrow[d, two heads, "\beta"] \\ J \arrow[r, hook, "i"] & M \end{tikzcd}$$

ψ is required to be injective.

Let P be a submodule of Q. By surjectivity of $P \twoheadrightarrow J$, it has basis $\{\lambda_j, \ldots \lambda_q\}$.

Next, consider

$$
\begin{tikzcd}
& P \arrow[rr, hook] \arrow[dl, dashed, "\lambda"'] \arrow[d, two heads, "\alpha"] & & Q \arrow[dl, dashed, "\exists ! s"'] \arrow[d, two heads, "\beta"] \\
N \arrow[r, two heads] & J & N \arrow[r, two heads] & M
\end{tikzcd}
$$

$Q \longrightarrow N$ is the basis mapping $\{\lambda_1, \ldots \lambda_q\} \longrightarrow$

$\{\lambda_1, \dots \lambda_n\}$. Hence, there must be a basis mapping

$\{\lambda_1, \dots \lambda_j, \dots, \lambda_n\} \longrightarrow \{\lambda_1 \dots \lambda_n\}$

We have already shown that $J \to N$ is injective. Hence, r and S are well-defined maps, and the diagram commutes.

<u>Ex</u> Given R heriditary show that every proj. resolution of R-module M has length 1.

The projectives of a heriditary R-module only contain submodules that are projective.

$$
\begin{tikzcd}
 & R \arrow[rr, "\varphi"] \arrow[d, "\alpha"] \arrow[dl, dashed, "r"'] & & S \arrow[d, "\beta"] \arrow[dl, dashed, "s"'] & \\
M \arrow[r, two heads, "p"'] & Q \arrow[r, hook, "\psi"] \arrow[rr, bend right, hook, "i"'] & M \arrow[r, two heads, "q"] & N \arrow[r] & 0
\end{tikzcd}
$$

Let N be a hereditary module, and let Q be a submodule. Let M be any other hereditary module. Since $M \to Q \to M$ is id_M, ψ is injective

We know: $\left.\begin{aligned} pr &= \alpha \\ qs &= \beta \end{aligned}\right\}$

ψ inj. α, β are surj.

To show: $\ker \beta = \operatorname{im} \varphi$.

$Q \longrightarrow N$ is the basis mapping

inj

$$\{\lambda_1 \ldots \lambda_q\} \longrightarrow \{\lambda_1 \ldots \lambda_q, \lambda_{q+1} \ldots \lambda_n\}$$

$R \xrightarrow{\alpha} Q$ is the mapping

$$\{\lambda_1 \ldots \lambda_r\} \longrightarrow \{\lambda_1 \ldots \lambda_q\}$$

$S \xrightarrow{\beta} N$ is the mapping

$$\{\lambda_1 \ldots \lambda_r \ldots \lambda_s\} \longrightarrow \{\lambda_1, \ldots \lambda_q \ldots \lambda_n\}$$

$R \xrightarrow{\varphi} S$ is the basis mapping

$\{\lambda_1 \ldots \lambda_r\} \longrightarrow$

$\{\lambda_1 \ldots \lambda_r \ldots \lambda_s\}$

Hence im φ has basis —

$\{\lambda_1 \ldots \lambda_r\}$

ker β has the same basis.

Hence, a proj. resolution of N is the s.e.s —→

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook] & S \arrow[r, two heads] & N \arrow[d] \\ & & & 0 \end{tikzcd}$$

Every other submodule would form an injection of this sort: $R \hookrightarrow S$ where R is the proj of the submod. and S is the proj. of the module! given

$Ext^2_R(M, N) = H^2 = 0$

$(L^2 Hom(N, -))(M)$

$$N \xrightarrow{d_\mu} M \xrightarrow{d_\varepsilon} N \longrightarrow M \longrightarrow 0$$

$H^2 = \dfrac{\ker d_\varepsilon}{\mathrm{im}\, d_\mu} = 0 \Rightarrow \underset{= 0}{Hom(M, N)}$

To show: $Ext^1(M, -) = 0$

$Ext^1(M, -) = Hom(M, -)$
$= 0$ from the above.

To show

$$N \xrightarrow{d_\mu} M \xrightarrow{d_\varepsilon} N$$

$$\frac{\ker d_\varepsilon}{\operatorname{im} d_\mu} = 0$$

$\Rightarrow \ker d_\varepsilon = \operatorname{im} d_\mu$

$\Rightarrow N \hookrightarrow M \twoheadrightarrow N$ is an S.E.S.

M, N are hereditary R-modules $\Rightarrow$ every submodule is proj.

Since M, N are submodules of themselves, trivially, every projective resolution of a projective module has $H^n = 0$.

<u>Ex</u> Given chain complex in additive cat —

$$G_\cdot - \quad C \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{pmatrix}B_1\\ D\end{pmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \varepsilon\end{pmatrix}} \begin{pmatrix}B_2\\ E\end{pmatrix} \xrightarrow{(\mu\ \nu)} F$$

Show that it is literally isomorphic to —

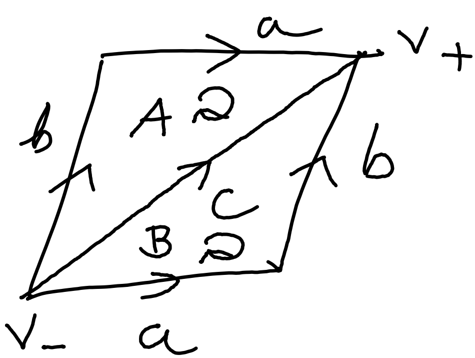
$$G_\cdot' - \quad C \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{pmatrix}B_1\\ D\end{pmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \varepsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{pmatrix}B_2\\ E\end{pmatrix} \xrightarrow{(0\ \nu)} F$$

if ϕ is an isomorphism.

Ex Given $T = \dfrac{X \times I}{(x,0) \sim (x,1)}$

and let $\phi_\theta : S^1 \longrightarrow S^1$ be a rotation by angle θ.

Find the homology of T in two different ways.



$\Delta^0 = \{v_-, v_+\}$

$\Delta^1 = 2\{a+b\}$

$= 2\{c\}$

$\Delta^2 = \{A, B\}$

$d_1 : \Delta^1 \longrightarrow \Delta^0$

$2\{a+b\} = 2\{c\} \xrightarrow{\ 0\ } \{v_-, v_+\}$

$\text{im } d_1 = 0$, $\text{Ker } d_1 = \mathbb{Z} \oplus \mathbb{Z}$

$$d_2 : \Delta^2 \longrightarrow \Delta^1$$

$$\{A, B\} \longrightarrow 2\{a+b\} = 2\{c\}$$

$$\{-c+a+b,\ c-a-b\}_0$$

$$\longrightarrow \cancel{2}\{-c+a+b,\ c-a-b,\ a+b\}$$

$$\left.\begin{aligned} \operatorname{Ker} d_0 &= \mathbb{Z} \\ \operatorname{im} d_0 &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} \operatorname{im} d_2 &= 0 \\ \operatorname{Ker} d_2 &= \mathbb{Z} \end{aligned}\right\}$$

$$\left.\begin{aligned} \operatorname{Ker} d_1 &= \mathbb{Z} \oplus \mathbb{Z} \\ \operatorname{im} d_1 &= 0 \end{aligned}\right\}$$

$$H_1 = \mathbb{Z} \oplus \mathbb{Z},\quad H_0 = \mathbb{Z}$$

$$H_2 = \mathbb{Z}$$

Doing it in cellular homology —

The cellular decomposition is —

$$\left.\begin{array}{ll} 1 & 0\text{-cell} \\ 2 & 1\text{-cells} \\ 1 & 2\text{-cell} \end{array}\right\}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

$$d_n(e^n_\beta) = \sum d_{\alpha\beta}(e^n_\alpha)$$

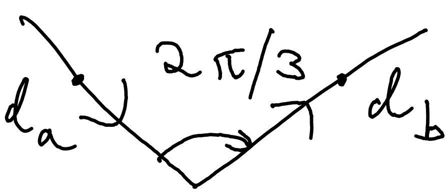
Hence, $H_0 = \mathbb{Z}$, $H_2 = \mathbb{Z}$

$H_1 = \mathbb{Z} \oplus \mathbb{Z}$.

<u>Ex</u> In the previous exercise, when $\theta = 2\pi/3$, compute homology.

$$\mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}^2 \xrightarrow{\mu} \mathbb{Z} \xrightarrow{0} 0$$

To find attaching maps ε and μ —



To compute degree for this $\frac{2\pi}{3}$ - wedge —

Reflection has local degrees $+1$ and -1 — it is rotation by 0 and the -1 corresponds to $\cos 0$ or $\cos \pi$. Hence, rotation by $\cos \frac{2\pi}{3} = -\frac{1}{2}$. The degrees add up in the wedge yielding $1 + \frac{1}{2} = \frac{3}{2}$

$$\mathbb{Z} \xrightarrow{3/2} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3/2 \\ 0 \end{pmatrix}} \mathbb{Z} \xrightarrow{0} 0$$

$$H_0 = \frac{\mathbb{Z}}{3/2\,\mathbb{Z}} = \frac{2\mathbb{Z}}{3\mathbb{Z}} \text{ or } \mathbb{Z}_{3/2}$$

$H_1 = \mathbb{Z} / 3/2\,\mathbb{Z} = \mathbb{Z}_{3/2}$. $H_2 = 0$

For a general θ —

$H_0 = \mathbb{Z}_{1-\cos\theta}$,

$H_1 = \mathbb{Z}_{1-\cos\theta}$,

$H_2 = 0$.

<u>Ex</u> Let $G \hookrightarrow H$ be a subgroup of a finite group. By restriction, every G-module is canonically a H-module. $\mathbb{Z}G$, $\mathbb{Z}H$ are the trivial group rings, considered as modules. Write